A Nominal Approach to Probabilistic Separation Logic

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ABSTRACT

Currently, there is a gap between the tools used by probability theorists and those used in formal reasoning about probabilistic programs. On the one hand, a probability theorist decomposes probabilistic state along the simple and natural product of probability spaces. On the other hand, recently developed probabilistic separation logics decompose state via relatively unfamiliar measure-theoretic constructions for computing unions of sigma-algebras and probability measures. We bridge the gap between these two perspectives by showing that these two methods of decomposition are equivalent up to a suitable equivalence of categories. Our main result is a probabilistic analog of the classic equivalence between the category of nominal sets and the Schanuel topos. Through this equivalence, we validate design decisions in prior work on probabilistic separation logic and create new connections to nominal-set-like models of probability.

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1 INTRODUCTION

Separation logic [41], now a standard tool for reasoning about programs with shared mutable state, grew out of Reynolds's Syntactic Control of Interference [40] — a substructural system for controlling the interaction of imperative program fragments. The basic ingredients for today's interpretations of separation logic connectives, present in the original model of Syntactic Control of Interference [34], can be seen as living in a category of functors known as the Schanuel topos, with noninterference defined in terms of the coproduct of finite sets. Over the years, this model has been reformulated to suit the needs of formal reasoning about imperative programs: modern models of separation logic live not in the Schanuel topos, but in categories more like **Set**, and separation is interpreted not by coproduct, but by algebraic structures such as partial commutative monoids (PCMs) [7, 21]. In particular, the now-standard model of separation logic in which separating conjunction splits stores into disjoint pieces is defined in terms of the partial function $\uplus$ sending a pair of disjoint stores to their union, giving rise to a PCM of stores. This shift in perspective is justified by a classic equivalence of categories:

Fact 1.1. The Schanuel topos **Sch** is equivalent to the category **Nom** of nominal sets, and the original coproduct-based model of separation in **Sch** corresponds to the standard union-based model in **Nom** across this equivalence.[1]

Today, there is a pressing need for syntactic control of *probabilistic* interference — that is, for establishing the *probabilistic independence* of program fragments. In response to this need, recent work has developed a number of probabilistic separation logics [3, 4, 6, 29], whose semantic models are given by PCMs made of probability-theoretic objects. Lilac [29] is a separation logic whose PCM-based model is particularly well-behaved: its notion of separation coincides with probabilistic independence [29, Lemma 2.5], and yields a frame rule identical to the standard one for store-based separation logics.

However, Lilac's PCM model does not match a probability theorist's intuition. One expects separation to be interpreted via a standard product of probability spaces [28], but Lilac interprets separation using *independent combination*: a partial binary operation on probability spaces constructed out of low-level set-theoretic operations on σ-algebras. Moreover, Lilac's model fixes up front an unconventional sample space — the space $[0,1]^\omega$ of infinite streams of real numbers in the interval, known as the Hilbert cube — and the soundness of Lilac's proof rules depends on various properties specific to it. These contrasts between Lilac's model and textbook probability raise a question: *how do we know Lilac provides a good notion of separation for probabilistic separation logic?*

We answer this question by showing Lilac's seemingly nonstandard independent combination is in fact *equivalent* to a probability theorist's product-based intuition of state decomposition. Our result is a probabilistic analog of Fact 1.1: just as the coproduct model of separation corresponds to the now-standard model based on $\uplus$ across an equivalence between the Schanuel topos and **Nom**, the probability theorist's intuitive product-based model of independence corresponds to Lilac's independent-combination-based model across an equivalence between a category of *enhanced measurable*

[1] For a good reference documenting this equivalence, see Pitts [38, §6.3].

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sheaves and a category of *absolutely continuous sets* (Theorem 4.35). Our contributions are as follows:

- We introduce *absolutely continuous sets*: just as nominal sets are sets equipped with an action by permutations of names, absolutely continuous sets are sets equipped with a continuous action by measurable permutations of the Hilbert cube.
- We prove analogs of the equivalence **Sch** $\simeq$ **Nom** for both discrete and continuous probability (Theorems 3.18 and 4.34). In particular, we show that the category $\mathbf{Set}^{\ll}$ of absolutely continuous sets is equivalent to a topos **EMS** of *enhanced measurable sheaves*: a probabilistic analog of the Schanuel topos.
- We show that $\mathbf{Set}^{\ll}$ provides a natural background category for a fragment of Lilac. Theorem 4.35 then shows that, by transporting across the equivalence $\mathbf{Set}^{\ll} \simeq \mathbf{EMS}$, Lilac's model corresponds to a model in **EMS** where separation arises naturally from product of probability spaces via Day convolution [7, 15, 35].

2 THE NOMINAL SITUATION

Our main result is a probabilistic analog of Fact 1.1 (Theorem 4.35). To set the stage, we first make Fact 1.1 a precise mathematical statement (Proposition 2.18). We devote this section to describing the necessary pieces in this comfortable setting; the material in this section is standard, but we will deviate occasionally from the usual presentation in order to focus on the aspects that are most relevant to our eventual probabilistic counterpart.

At its core, Fact 1.1 states that two distinct approaches to modelling store-separation are equivalent. To illustrate this fact we will study a tiny separation logic consisting of propositions P, Q about integer-valued stores:

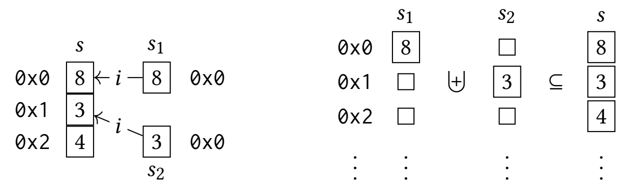
$$P, Q ::= x \mapsto i \mid \mathsf{True} \mid P * Q. \qquad (\textsc{TinySep})$$

TinySep propositions are well-formed according to a judgment $\Gamma \vdash P$ defined as usual: a context Γ is a set of logical variables x, and $\Gamma \vdash P$ if Γ contains the variables used in P. Fact 1.1 asserts the equivalence of two different models for TinySep:

Model 1: separation as coproduct. In this model, a store consists of two components: (1) a *shape* L given as a finite set of available locations (i.e., memory addresses), and (2) a *valuation* $s : L \rightharpoonup \mathbb{Z}$, a partial function assigning values to a subset of the shape. An example is shown in Figure 1a; the store s has shape $\{\mathtt{0x0}, \mathtt{0x1}, \mathtt{0x2}\}$, and the valuation maps $\mathtt{0x0} \mapsto 8$ and so on. Under this model, the meaning of a proposition depends on the shape L: the interpretation of a proposition $\Gamma \vdash P$ has form $[\![\Gamma \vdash P]\!]_1^L : (\Gamma \to L) \to \mathcal{P}(L \rightharpoonup \mathbb{Z})$, associating each substitution $\gamma : \Gamma \to L$ to the set $[\![\Gamma \vdash P]\!]_1^L(\gamma)$ of L-shaped valuations satisfying P.

Under this interpretation, we define $s \in [\![\mathsf{True}]\!]_1^L(\gamma)$ always and $s \in [\![x \mapsto i]\!]_1^L(\gamma)$ if and only if $s(\gamma(x)) = i$. Separating conjunction is defined via the coproduct of store shapes: $P_1 * P_2$ holds of an L-shaped valuation s if and only if there are valuations s_1 of shape L_1 and s_2 of shape L_2, and an injective function $i : L_1 + L_2 \hookrightarrow L$ embedding the coproduct $L_1 + L_2$ into L such that s_1 satisfies P_1 and s_2 satisfies P_2 and s_1, s_2 embed into s along i. This situation is visualized in Figure 1a. For example,

$$s \in [\![(x \mapsto 8) * (y \mapsto 3)]\!]_1^{\{\mathtt{0x0},\mathtt{0x1},\mathtt{0x2}\}}(\{x \mapsto \mathtt{0x0}, y \mapsto \mathtt{0x1}\})$$



(a) Model 1: coproduct. **(b) Model 2: union.**

Figure 1: Visualizing separation in Model 1 and Model 2.

is witnessed by setting s_1 to the $\{\mathtt{0x0}\}$-shaped valuation $\{\mathtt{0x0} \mapsto 8\}$ and s_2 to the $\{\mathtt{0x0}\}$-shaped valuation $\{\mathtt{0x0} \mapsto 3\}$ and $i : \{\mathtt{0x0}\} + \{\mathtt{0x1}\} \hookrightarrow \{\mathtt{0x0}, \mathtt{0x1}\}$ to the injection defined by $i(\mathrm{inl}(\mathtt{0x0})) = \mathtt{0x0}$ and $i(\mathrm{inr}(\mathtt{0x0})) = \mathtt{0x1}$, where $\mathrm{inl} : L_1 \to L_1 + L_2$ and $\mathrm{inr} : L_2 \to L_1 + L_2$ are the coproduct injections.

Model 2: separation as union. In this model, one fixes upfront a "universal store shape" into which all store shapes can be embedded. Any countably-infinite set will do; we choose the natural numbers $\mathbb{N}$. A store is a partial function $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ defined on finitely many values of its domain, and a proposition $\Gamma \vdash P$ denotes a function $[\![\Gamma \vdash P]\!]_2 : (\Gamma \to \mathbb{N}) \to \mathcal{P}(\mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z})$. The interpretations of True and $x \mapsto i$ are as in the shape-indexed model: $s \in [\![\mathsf{True}]\!]_2(\gamma)$ always and $s \in [\![x \mapsto i]\!]_2(\gamma)$ if and only if $s(\gamma(x)) = i$. Separating conjunction is defined via union of stores: a store s is in $[\![P_1 * P_2]\!]_2(\gamma)$ if and only if there exist disjoint stores s_1 and s_2 with $s_1 \uplus s_2 \subseteq s$ such that s_1 is in $[\![P_1]\!]_2(\gamma)$ and s_2 is in $[\![P_2]\!]_2(\gamma)$. Figure 1b visualizes an example: $s \in [\![(x \mapsto 8) * (y \mapsto 3)]\!]_2 \{x \mapsto \mathtt{0x0}, y \mapsto \mathtt{0x1}\}$ holds because s_1 and s_2 have a union contained in s and s_1 satisfies $x \mapsto 8$ and s_2 satisfies $y \mapsto 3$.

Relating the two models. Model 1 and Model 2 are equivalent by Fact 1.1. The equivalence is based on the following idea. Every store shape L can be encoded as a finite subset of $\mathbb{N}$ via a suitable pair of functions $\mathrm{enc}_L : L \to \mathbb{N}$ and $\mathrm{dec}_L : \mathbb{N} \rightharpoonup L$. Choosing an arbitrary such pair $(\mathrm{enc}_L, \mathrm{dec}_L)$ for every L allows translating Model 1 into Model 2 in a bijective way: an L-shaped store $s : L \rightharpoonup \mathbb{Z}$ corresponds to a finite partial function $s \circ \mathrm{dec}_L : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$, and a Model-1-substitution $\gamma : \Gamma \to L$ corresponds to a Model-2-substitution $\mathrm{enc}_L \circ \gamma : \Gamma \to \mathbb{N}$. Via these translations, it holds that $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ if and only if $s \circ \mathrm{dec}_L \in [\![\Gamma \vdash P]\!]_2(\mathrm{enc}_L \circ \gamma)$ for all L-shaped valuations s, propositions $\Gamma \vdash P$, and substitutions $\gamma : \Gamma \to L$, so both models induce the same notion of store-satisfaction.

This equivalence should seem plausible enough given how tiny TinySep is. What is remarkable about Fact 1.1 is that this equivalence continues to hold when the interpretations $[\![-]\!]_1^{(-)}$ and $[\![-]\!]_2$ are extended to include all the usual features of separation logic, including separating implication $-\!\!*$, the intuitionistic connectives $\wedge, \vee, \to$, False, quantification at both first-order and higher type, quantification over propositions, predicates defined by structural recursion, and so on. In short, the semantic domains of Model 1 are equivalent in expressive power to those of Model 2.

Rather than laboriously verifying one by one that the standard interpretations of each of these features coincide, Fact 1.1 establishes a general result. The key is to place Model 1 and Model 2 into the context of suitable categories that bring out their essential structure. Model 1 naturally lives in a category **Sch** called the *Schanuel topos*: the interpretation $[\![\Gamma \vdash P]\!]_1^{(-)}$ of a proposition P defines a **Sch**-morphism from a **Sch**-object representing Γ-substitutions to a **Sch**-object representing store-predicates. Model 2 naturally lives in the category **Nom** of *nominal sets* [38]: the interpretation $[\![\Gamma \vdash P]\!]_2$ defines a **Nom**-morphism from a nominal set of Γ-substitutions to a nominal set of store-predicates. Having placed Model 1 and Model 2 into suitable background categories, Fact 1.1 follows from a classic theorem: the categories **Sch** and **Nom** are known to be equivalent [38, §6.3], and inspecting the proof of this equivalence shows that the functor $\mathbf{Sch} \to \mathbf{Nom}$ witnessing it sends Model 1 to Model 2 via the construction involving $(\mathrm{enc}_L, \mathrm{dec}_L)$.

The rest of this section is devoted to filling in the details of this category-theoretic setup. First we will describe how Model 1 lives in **Sch** and Model 2 lives in **Nom**. Then we will highlight the essential properties of this setup that make the equivalence $\mathbf{Sch} \simeq \mathbf{Nom}$ possible, and how Model 1 and Model 2 are instances of the same structure across this equivalence; Theorem 4.35 relies crucially on identifying analogous properties in the probabilistic setting.

2.1 Model 1 in the Schanuel topos

In this section we describe how Model 1 of Section 2 naturally lives in the Schanuel topos **Sch**. The benefit of this is that it makes the invariants maintained by $[\![-]\!]_1^{(-)}$ explicit: the category **Sch** is such that all constructions that make categorical sense — i.e., are well-defined as objects and morphisms of **Sch** — are forced to preserve all invariants. The invariants in this case are the following principles one intuitively expects to hold when reasoning about stores:

- *Extension*: propositions should continue to hold when new locations are introduced (such as when declaring a local variable or allocating a reference). More precisely, if $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ for some L-shaped valuation s and substitution $\gamma : \Gamma \to L$, and if L is a subset of some extended set of locations L', then it should hold that $s \in [\![\Gamma \vdash P]\!]_1^{L'}(\gamma)$, where we have implicitly coerced s into an L'-shaped valuation and γ into an L'-shaped substitution $\Gamma \to L'$ along the inclusion $L \subseteq L'$. [2]
- *Renaming*: propositions should be stable under renaming of locations. More precisely, if $s \in [\![\Gamma \vdash P]\!]_1^L(\gamma)$ for some L-shaped valuation s and substitution $\gamma : \Gamma \to L$, and if f is a bijective function $L \to L'$, then it should hold that $s \circ f^{-1} \in [\![\Gamma \vdash P]\!]_1^{L'}(f \circ \gamma)$.
- *Restriction*: the truth of a proposition should not depend on any unused locations. For example, suppose a proposition P holds of the $\{\ell_1, \ell_2\}$-shaped valuation $\{\ell_1 \mapsto 1\}$, which does not use the location ℓ_2. Then P should also hold of $\{\ell_1 \mapsto 1\}$ considered as an $\{\ell_1\}$-shaped valuation.

As basic principles of store-based reasoning, it is crucial that these invariants are preserved by the basic separation logic connectives: if P and Q satisfy Extension, Renaming, and Restriction, then their separating conjunction $P * Q$, separating implication $P \mathrel{-\!\!*} Q$, conjunction $P \wedge Q$, and implication $P \to Q$ should as well.

A general strategy for preserving invariants like this is to work with **Set**-valued functors out of a category C capturing them. Such functors are very well-behaved: in particular, many subcategories of the functor category $[C^{\mathrm{op}}; \mathbf{Set}]$, called *categories of sheaves on* C, are automatically cartesian closed, and can be used to quickly obtain invariant-preserving interpretations of logical connectives. Placing Model 1 into the Schanuel topos **Sch** is an instance of this idea. The Schanuel topos is a particular subcategory of $[C^{\mathrm{op}}; \mathbf{Set}]$, where C is chosen so that functors $C^{\mathrm{op}} \to \mathbf{Set}$ capture Extension and Renaming, consisting only of functors that are *atomic sheaves* in order to capture Restriction. We build up to this model in steps.

2.1.1 The base category C. Essentially, Extension says propositions should be stable under subset-inclusions $L \subseteq L'$ and Renaming says they should be stable under bijections. These two invariants can be packaged into a *category of store shapes*:

Definition 2.1. Let **Shp** be the category whose objects are finite sets L and whose morphisms from L to M are functions $M \to L$ definable by composing subset-inclusions and bijections.

Note the direction $M \to L$ is the reverse of what one might expect; this is because we will consider contravariant functors on **Shp**. Intuitively, there is a morphism $M \to L$ if L is a "smaller" shape than M. Since every composite of subset-inclusions and bijections is an injective function, and every injective function is bijective onto its image, the category **Shp** has a simple abstract description:

Proposition 2.2. The category **Shp** is equal to $\mathbf{Inj}_{<\omega}^{\mathrm{op}}$, where $\mathbf{Inj}_{<\omega}$ is the category of injective functions between finite sets.

With **Shp** in hand, functors $\mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ (equivalently, functors $\mathbf{Inj}_{<\omega} \to \mathbf{Set}$) model Extension- and Renaming-invariant concepts. In particular, there is a functor modelling stores:

Definition 2.3 (Store functor)**.** The *store functor* $\mathrm{S} : \mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ is a functor that sends a finite set L to the set of all L-shaped valuations and a $\mathbf{Inj}_{<\omega}$-morphism $i : L \hookrightarrow M$ to a function coercing $\mathrm{S}(M)$ into $\mathrm{S}(L)$ defined by $\mathrm{S}(i)(L, s) = (M, s')$, where s' is the valuation $M \rightharpoonup \mathbb{Z}$ defined by $s'(m) = s(l)$ iff $m = i(l)$ for some l in L.

The action of S on **Shp**-morphisms captures the operations that we expect to be invariant under: if i is a subset inclusion $L \subseteq L'$, then $\mathrm{S}(i)$ coerces L-shaped stores into L'-shaped stores as in the description of Extension, and if f is a bijective function $L \to L'$, then $\mathrm{S}(f)$ sends an L-shaped valuation s to an L'-shaped valuation $s \circ f^{-1}$ as in the description of Renaming.

2.1.2 Using sheaves to capture Restriction. Recall the example used to illustrate Restriction: if a proposition holds of the $\{\ell_1, \ell_2\}$-shaped valuation $\{\ell_1 \mapsto 1\}$, then it should also hold of $\{\ell_1 \mapsto 1\}$ considered as an $\{\ell_1\}$-shaped valuation. We say that $\{\ell_1 \mapsto 1\} \in \mathrm{S}\{\ell_1, \ell_2\}$ *restricts to* $\{\ell_1 \mapsto 1\} \in \mathrm{S}\{\ell_1\}$ *along* i, where i is the subset-inclusion $\{\ell_1\} \subseteq \{\ell_1, \ell_2\}$. This is an instance of a more general property satisfied by the functor S:

Proposition 2.4. Let $i : L \hookrightarrow M$ be an injective function between finite sets L and M, and $s \in \mathrm{S}(M)$ an M-shaped valuation. If $\mathrm{dom}(s) \subseteq \mathrm{im}(i)$, then there exists a unique L-shaped valuation $s' \in \mathrm{S}(L)$, the *restriction of* s *along* i, such that $\mathrm{S}(i)(s') = s$.

[2] This makes the separation logic affine rather than linear; we will restrict our attention to affine separation logics in this paper, as Lilac is affine and our main goal is to obtain models for it.

Proposition 2.4 can be expressed more abstractly:

Definition 2.5. Let F be a functor $\mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ and $i : L \hookrightarrow M$ a $\mathbf{Inj}_{<\omega}$-morphism. An element y of $F(M)$ is *restrictable along* i if for all $\mathbf{Inj}_{<\omega}$-objects N and $\mathbf{Inj}_{<\omega}$-morphisms $j, k : M \to N$ with $j \circ i = k \circ i$ it holds that $F(j)(y) = F(k)(y)$.

Definition 2.6. A functor $F : \mathbf{Shp}^{\mathrm{op}} \to \mathbf{Set}$ *has a restriction operation* if for all $\mathbf{Inj}_{<\omega}$-morphisms $i : L \hookrightarrow M$ and elements y of $F(M)$ that are restrictable along i, there exists a unique $x \in F(L)$, called the *restriction of* y *along* i, such that $y = F(i)(x)$.

With these definitions in hand, one can show Proposition 2.4 is equivalent to S having a restriction operation. Functors with a restriction operation have a special name: they are called *atomic sheaves on* $\mathbf{Shp}$ [30, Lemma III.4.2]. The Schanuel topos $\mathbf{Sch}$ is the full subcategory of $[\mathbf{Shp}^{\mathrm{op}}; \mathbf{Set}]$ consisting of atomic sheaves.

In these new terms, Proposition 2.4 says S is an atomic sheaf on $\mathbf{Shp}$, and so an object of $\mathbf{Sch}$. Just as S captures the concept of stores as shape-indexed valuations, there are other atomic sheaves for each of the other concepts used to define Model 1:

Proposition 2.7. The following are objects of $\mathbf{Sch}$:

- The constant functor Prop sending every object of $\mathbf{Shp}$ to the set $\{\top, \bot\}$ and every morphism of $\mathbf{Shp}$ to the identity function.
- The functor Loc of locations, defined by $\mathrm{Loc}(L) = L$ on objects of $\mathbf{Shp}$ and $\mathrm{Loc}(i : L \hookrightarrow L')(l : L) = i(l)$ on $\mathbf{Inj}_{<\omega}$-morphisms $i : L \hookrightarrow L'$.
- The functor Loc^{Γ} of Γ-substitutions, which maps objects L to the set of all substitutions $L \to \Gamma$, and action on $\mathbf{Inj}_{<\omega}$-morphisms inherited pointwise from Loc.

With these sheaves in hand, one can show Model 1 lives in $\mathbf{Sch}$:

Proposition 2.8. If $\Gamma \vdash P$ then the L-indexed family of functions

$$\left(\llbracket \Gamma \vdash P \rrbracket_1^L : (\Gamma \to L) \to \mathcal{P}(L \rightharpoonup \mathbb{Z})\right)_{L \in \mathbf{Shp}}$$

is natural in L, so defines a morphism $\mathrm{Loc}^{\Gamma} \to \mathrm{Prop}^{\mathsf{S}}$ in $\mathbf{Sch}$, where $\mathrm{Prop}^{\mathsf{S}}$ is the exponential guaranteed to exist because $\mathbf{Sch}$ is cartesian closed by virtue of being a category of sheaves. Moreover, every morphism of this type satisfies Extension, Renaming, and Restriction.

2.2 Model 2 in nominal sets

We now turn to the other side of the equivalence given by Fact 1.1: the category of nominal sets $\mathbf{Nom}$, and how it naturally houses Model 2 of Section 2, in which separation is defined via union of finite partial functions on $\mathbb{N}$.

Just as $\mathbf{Sch}$ is a category capturing the invariants implicitly maintained by Model 1, $\mathbf{Nom}$ is a category capturing the invariants implicitly maintained by Model 2. In this case, the invariants are:

- *Permutation*: propositions should be stable under permuting locations. If $s \in \llbracket \Gamma \vdash P \rrbracket_2 (\gamma)$ for some store $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ and substitution $\gamma : \Gamma \to \mathbb{N}$, and $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation of finitely-many natural numbers, then it should hold that $s \circ \pi \in \llbracket \Gamma \vdash P \rrbracket_2 (\pi^{-1} \circ \gamma)$.
- *Finiteness*: more subtly, stores and substitutions can only mention finitely-many locations $n \in \mathbb{N}$; this models the fact that physical stores are necessarily finite, and ensures that one always has the ability to allocate fresh locations.

To capture Permutation, the objects of $\mathbf{Nom}$ are sets equipped with an action by a group of permutations to be invariant under. Specifically, let S_ω be the group of permutations of finitely-many natural numbers: elements of S_ω are bijective functions $\pi : \mathbb{N} \to \mathbb{N}$ such that there exists some $n \in \mathbb{N}$ with $\pi(m) = m$ for all $m \geq n$. An S_ω*-set* is a set X equipped with a right action by S_ω: a function $(\cdot) : X \times S_\omega \to X$ satisfying $x \cdot 1 = x$ and $x \cdot (\pi\sigma) = (x \cdot \pi) \cdot \sigma$ for all $x \in X$ and $\pi, \sigma \in S_\omega$. There is an S_ω-set $\overline{\mathsf{S}}$ of stores, whose group action says what it means to permute the locations in a store:[3]

Definition 2.9. Let $\overline{\mathsf{S}}$ be the S_ω-set of stores $s : \mathbb{N} \xrightarrow{\text{fin}} \mathbb{Z}$ with action $s \cdot \pi = s \circ \pi$.

A morphism of S_ω-sets $(X, \cdot_X) \to (Y, \cdot_Y)$ is an *equivariant function*: a function $f : X \to Y$ satisfying $f(x \cdot_X \pi) = f(x) \cdot_Y \pi$ for all $x \in X$ and $y \in Y$ and $\pi \in S_\omega$. This captures invariance under Permutation: S_ω-morphisms $\overline{\mathsf{S}} \to \overline{\mathrm{Prop}}$, where $\overline{\mathrm{Prop}}$ is the S_ω-set $\{\top, \bot\}$ with trivial action $p \cdot \pi = p$, are the permutation-invariant predicates on stores.

To capture Finiteness, the category $\mathbf{Nom}$ is a full subcategory of the category of S_ω-sets consisting of those S_ω-sets $(X, \cdot_X)$ in which every $x \in X$ only uses finitely many locations. The concept of "using" a location is made precise by looking at stabilizer subgroups: if $x \cdot \pi = x$ (i.e., π is in the stabilizer of x) then x can only "use" those locations fixed by π. An element x uses finitely many locations if its stabilizer is *open* for a suitable topology:

Definition 2.10 (Topology on S_ω)**.** A subset U of S_ω is *open* if for every π in U there exists a finite subset A of $\mathbb{N}$ such that $\pi \in \mathrm{Fix}\, A \subseteq U$, where $\mathrm{Fix}\, A$ is the subgroup of S_ω-permutations π that fix every element of A; i.e., $\pi(a) = a$ for all a in A.

Intuitively, a stabilizer subgroup $\mathrm{Stab}\, x$ is open if every π stabilizing x does so for some "finite reason" A: there is some subset A fixed by π such that any other permutation π' fixing A also stabilizes x. Nominal sets are S_ω-sets with open stabilizers [37, §6.2]:

Definition 2.11. A *nominal set* is a S_ω-set $(X, \cdot)$ such that for every x in X the stabilizer subgroup $\mathrm{Stab}\, x$ is open. $\mathbf{Nom}$ is the category of nominal sets and equivariant functions.

For example, $\overline{\mathsf{S}}$ is a nominal set: if s is a store with $s \circ \pi = s$, then π fixes the finite set $\mathrm{dom}(s)$, and moreover every permutation fixing $\mathrm{dom}(s)$ fixes s, so $\mathrm{dom}(s) \subseteq \mathrm{Stab}\, x$ and $\mathrm{Stab}\, s$ is open. There are nominal sets capturing each of the other concepts used in Model 2:

Proposition 2.12. The following are objects of $\mathbf{Nom}$:

- The S_ω-set $\overline{\mathrm{Prop}}$ of propositions
- The S_ω-set $\overline{\mathrm{Loc}}$ of locations $\mathbb{N}$ with action $x \cdot \pi = \pi^{-1}(x)$.
- The S_ω-set $\overline{\mathrm{Loc}}^{\Gamma}$ of Γ-substitutions $\gamma : \Gamma \to \mathbb{N}$ with action defined by $\gamma \cdot \pi = \pi^{-1} \circ \gamma$.

With these in hand, one can show Model 2 lives in $\mathbf{Nom}$:

Proposition 2.13. If $\Gamma \vdash P$ then the function $\llbracket \Gamma \vdash P \rrbracket_2$ is a morphism $\overline{\mathrm{Loc}}^{\Gamma} \to \overline{\mathrm{Prop}}^{\overline{\mathsf{S}}}$ in $\mathbf{Nom}$, and every morphism of this type satisfies Permutation and Finiteness.

[3] In general, we will overline objects of $\mathbf{Nom}$ to distinguish them from their $\mathbf{Sch}$-counterparts.

2.3 The equivalence

This section sketches the classic equivalence **Sch** $\simeq$ **Nom** and how Models 1 and 2 correspond across it. We will not be concerned so much with the details of this particular equivalence, but rather with highlighting the key properties of **Sch** and **Nom** that make it possible — Theorem 4.35 relies on identifying analogous properties in the probabilistic setting.

In Section 2 we sketched the correspondence between Model 1 and Model 2, based on the idea that every store shape L can be encoded as a finite set of natural numbers via a pair of functions $(\mathrm{enc}_L, \mathrm{dec}_L)$. This idea also forms the basis for the equivalence **Sch** $\simeq$ **Nom**. In the language of Section 2.1, every enc_L encodes the object L of **Shp** as a subset $\mathrm{im}(\mathrm{enc}_L)$ of $\mathbb{N}$. This encoding extends to morphisms of **Shp**: every **Shp**-morphism $M \to L$, equivalently an injective function $f : L \hookrightarrow M$, can be encoded as a permutation $\pi \in S_\omega$ that sends $\mathrm{im}(\mathrm{enc}_L)$ to $\mathrm{im}(\mathrm{enc}_M)$. More precisely,

Proposition 2.14 (Homogeneity [38, L1.14])**.** Let L, M be finite sets and enc_L and enc_M injective functions $L \hookrightarrow \mathbb{N}$ and $M \hookrightarrow \mathbb{N}$. For any injective function $i : L \hookrightarrow M$, there exists $\pi \in S_\omega$ such that $\pi \circ \mathrm{enc}_L = \mathrm{enc}_M \circ i$, making the following square commute:

$$\begin{tikzcd} \mathbb{N} \arrow[r, dashed, "\pi"] & \mathbb{N} \\ L \arrow[u, hook, "\mathrm{enc}_L"] \arrow[r, hook, "i"'] & M \arrow[u, hook, "\mathrm{enc}_M"'] \end{tikzcd}$$

Furthermore, the relationships between encoded store shapes $\mathrm{im}(\mathrm{enc}_L)$ are faithfully captured by relationships between subgroups of S_ω:

Proposition 2.15 (Correspondence)**.** For any two store shapes L and M, it holds that $\mathrm{Fix}(\mathrm{im}(\mathrm{enc}_L)) \subseteq \mathrm{Fix}(\mathrm{im}(\mathrm{enc}_M))$ if and only if $\mathrm{im}(\mathrm{enc}_L) \supseteq \mathrm{im}(\mathrm{enc}_M)$.

Homogeneity and Correspondence together give the equivalence **Sch** $\simeq$ **Nom**. For details, see MacLane and Moerdijk [30, Theorem III.9.2]. With this equivalence in hand, we are finally in a position to make Fact 1.1 precise. Abbreviating Loc^Γ as $[\![\Gamma]\!]$ and the exponential $\mathrm{Prop}^{\mathrm{S}}$ as Pred, Proposition 2.8 shows that the Hom-set $\mathbf{Sch}([\![\Gamma]\!], \mathrm{Pred})$ serves as a semantic domain for Model-1 interpretations of TinySep propositions in context Γ. Analogously, Proposition 2.13 shows that $\mathbf{Nom}(\overline{[\![\Gamma]\!]}, \overline{\mathrm{Pred}})$ serves as a semantic domain for Model 2, where $\overline{[\![\Gamma]\!]}$ is $\overline{\mathrm{Loc}}^\Gamma$ and $\overline{\mathrm{Pred}}$ the exponential $\overline{\mathrm{Prop}}^{\overline{\mathrm{S}}}$. The next proposition establishes that these semantic domains correspond across **Sch** $\simeq$ **Nom**:

Proposition 2.16. Across the equivalence **Sch** $\simeq$ **Nom**, the sheaf S corresponds to the nominal set $\overline{\mathrm{S}}$, Prop to $\overline{\mathrm{Prop}}$, Loc to $\overline{\mathrm{Loc}}$, $[\![\Gamma]\!]$ to $\overline{[\![\Gamma]\!]}$, Pred to $\overline{\mathrm{Pred}}$, and $\mathbf{Sch}([\![\Gamma]\!], \mathrm{Pred})$ to $\mathbf{Nom}(\overline{[\![\Gamma]\!]}, \overline{\mathrm{Pred}})$.

It remains to show that Model 1 intepretations $[\![\Gamma \vdash P]\!]_1^{(-)}$ correspond to their Model 2 counterparts $[\![\Gamma \vdash P]\!]_2$. This is straightforward when P is True or $x \mapsto i$; the interesting case is the separating conjunction $P_1 * P_2$. One could show $[\![P_1 * P_2]\!]_1^{(-)}$ corresponds to $[\![P_1 * P_2]\!]_2$ by unwinding definitions and showing, via a careful calculation, that they correspond across the functor **Sch** $\to$ **Nom** witnessing the equivalence. But Fact 1.1 is far more general. The idea is to use the *internal language* of **Sch**: as a category of sheaves, any construction in higher-order logic can be interpreted in **Sch** [30, VI.7.1]. In this internal language, types denote sheaves and functions denote natural transformations, and Model 1's separating conjunction can be defined as $[\![\Gamma \vdash P_1 * P_2]\!]_1^{(-)} = [\![\Gamma \vdash P_1]\!]_1^{(-)} \circledast [\![\Gamma \vdash P_2]\!]_1^{(-)}$, where $\circledast$ is a special **Sch**-morphism denoting separating conjunction in the internal language of **Sch**. The meaning of $\circledast$ can be described by conveniently using the higher-order logic of **Sch**:

$$(\circledast) : \mathrm{Pred}^{[\![\Gamma]\!]} \times \mathrm{Pred}^{[\![\Gamma]\!]} \to \mathrm{Pred}^{[\![\Gamma]\!]}$$

$$(f_1 \circledast f_2)(\gamma : [\![\Gamma]\!])(s : \mathrm{S}) = \begin{pmatrix} \exists\, s_1\, s_2 : \mathrm{S}.\ s_1 \bullet s_2 \text{ defined} \wedge \\ s_1 \bullet s_2 \sqsubseteq s \wedge f_1\,\gamma\,s_1 \wedge f_2\,\gamma\,s_2 \end{pmatrix} \tag{1}$$

This definition is made of the following key ingredients:

- A symbol $\sqsubseteq$, which in the internal language looks like an ordering relation on stores, and externally denotes a suitable natural transformation $\mathrm{S} \times \mathrm{S} \to \mathrm{Prop}$.
- A symbol $\bullet$, which internally looks like a partial function combining stores, and externally denotes a natural transformation $\mathrm{S}^2_\perp \to \mathrm{S}$, where $\mathrm{S}^2_\perp$ is a subobject $i : \mathrm{S}^2_\perp \hookrightarrow \mathrm{S} \times \mathrm{S}$ of the sheaf $\mathrm{S} \times \mathrm{S}$ of pairs of stores carving out the domain on which $\bullet$ is defined.

The ordering $\sqsubseteq$ is the natural transformation $(\sqsubseteq) : \mathrm{S} \times \mathrm{S} \to \mathrm{Prop}$ defined by $(s_1 \sqsubseteq_L s_2) = \top$ if and only if s_1 is a subvaluation of s_2. The combining operation $\bullet$ is a natural transformation $\bullet : \mathrm{S}^2_\perp \to \mathrm{S}$. Its domain $\mathrm{S}^2_\perp$ is a sheaf defined in terms of the coproduct of finite sets. Each $\mathrm{S}^2_\perp(L)$ is a set consisting of pairs of L-shaped valuations that "factor through" a coproduct $L_1 + L_2$ along some $i : L_1 + L_2 \hookrightarrow L$:

$$\mathrm{S}^2_\perp(L) = \begin{array}{l} \{\, (\mathrm{S}(i \circ \mathrm{inl})(s_1), \mathrm{S}(i \circ \mathrm{inr})(s_2)) \\ \mid L_1, L_2 \in \mathbf{Shp}, s_1 \in \mathrm{S}(L_1), s_2 \in \mathrm{S}(L_2), i : L_1 + L_2 \hookrightarrow L \,\} \end{array}$$

The morphism $\bullet$ sends each pair $(\mathrm{S}(i \circ \mathrm{inl})(s_1), \mathrm{S}(i \circ \mathrm{inr})(s_2))$ of separated stores to the combined store $\mathrm{S}(i)[s_1, s_2]$, where the valuation $[s_1, s_2]$ of type $L_1 + L_2 \to \mathbb{Z}$ is the unique one defined by $[s_1, s_2] \circ \mathrm{inl} = s_1$ and $[s_1, s_2] \circ \mathrm{inr} = s_2$. Each $\mathrm{S}^2_\perp(L)$ is a subset of $(\mathrm{S} \times \mathrm{S})(L)$, and collecting the canonical subset-inclusions into an L-indexed family gives a monic natural transformation $i : \mathrm{S}^2_\perp \hookrightarrow \mathrm{S} \times \mathrm{S}$.

In the internal language, $\bullet$ looks like a partial function that is associative and commutative and monotone with respect to $\sqsubseteq$, with unit the natural transformation $\mathrm{emp} : 1 \to \mathrm{S}$ sending every store shape L to the empty valuation on L. Together, the tuple $(\sqsubseteq, \mathrm{S}^2_\perp, i, \bullet, \mathrm{emp})$ packages up the ingredients needed to model separation logic in **Sch** into a *resource monoid* internal to **Sch**:

Definition 2.17. A *resource monoid* [21] is a poset $(R, \sqsubseteq)$ with a least element $\perp$ and a monotone partial function $(\cdot) : R \times R \rightharpoonup R$ such that $(R, \cdot, \perp)$ forms a partial commutative monoid.[4]

We can similarly construct a resource monoid in **Nom**. There is an equivariant function $(\sqsubseteq) : \overline{\mathrm{S}} \times \overline{\mathrm{S}} \to \overline{\mathrm{Prop}}$ sending a pair (s_1, s_2) of finite partial functions on $\mathbb{N}$ to $\top$ if and only if $s_1 \subseteq s_2$, with least element $\overline{\mathrm{emp}}$ the empty finite partial function. There is a nominal set $\overline{\mathrm{S}}^2_\perp$ of separated stores: the set

$$\{(s_1, s_2) \mid s_1, s_2 \in \overline{\mathrm{S}}, \mathrm{dom}(s_1) \cap \mathrm{dom}(s_2) = \emptyset\}$$

of pairs of stores with disjoint domain, and pointwise action. Both the canonical inclusion $\overline{i} : \overline{\mathrm{S}}^2_\perp \hookrightarrow \overline{\mathrm{S}} \times \overline{\mathrm{S}}$ and the function $(\overline{\bullet}) : \overline{\mathrm{S}}^2_\perp \to \overline{\mathrm{S}}$ sending a pair (s_1, s_2) of disjoint stores to their union

[4] In this paper we are concerned with affine models of separation logic, and so consider an affine variant of the resource monoids defined in Galmiche et al. [21]. Our definition is closest in spirit to the affine PDMs sketched there.

$s_1 \uplus s_2$ are equivariant, hence morphisms in **Nom**. Finally, $\bar{\bullet}$ is monotone in $\bar{i}$ and has unit $\overline{\mathrm{emp}}$ so $(\overline{\sqsubseteq}, \overline{\mathrm{S}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ forms a resource monoid internal to **Nom**, and reinterpreting Eq. (1) inside **Nom** with $(\overline{\sqsubseteq}, \overline{\mathrm{S}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ in place of $(\sqsubseteq, \mathrm{S}^2_\perp, i, \bullet, \mathrm{emp})$ yields Model 2's separating conjunction. The following proposition, connecting the two resource monoids, makes Fact 1.1 precise:

Proposition 2.18. The resource monoid $(\sqsubseteq, \mathrm{S}^2_\perp, i, \bullet, \mathrm{emp})$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathrm{S}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ across the equivalence **Sch** $\simeq$ **Nom**.

We are at last ready to appreciate the full power of this fact. First, it shows $[\![P_1 * P_2]\!]_1^{(-)}$ corresponds to $[\![P_1 * P_2]\!]_2$: both arise from the same internal-language definition, up to the replacement of types and function symbols following Propositions 2.16 and 2.18. Next, since the separating implication $-\!\!*$ and all intuitionistic connectives can be defined similarly using the internal language, they must correspond as well; this extends the equivalence of Models 1 and 2 from TinySep to all standard separation logic connectives. More generally, Fact 1.1 says that any construction in higher-order logic that only uses the types and functions of Propositions 2.16 and 2.18 corresponds across the equivalence **Sch** $\simeq$ **Nom**.

3 THE DISCRETE CASE

Theorem 4.35 imports quite a bit of measure theory in order to support continuous probability. To describe the key ideas, we temporarily set the measure theory aside by first presenting in detail a version of Theorem 4.35 adapted to discrete probability.

The structure of this section is completely analogous to Section 2. We first present two different probabilistic separation logics: one where separation is defined via the product of sample spaces, and a second based on Li et al. [29] where separation is defined via *independent combination*. Then, we will show how separation-as-product naturally lives in a category $\mathbf{EMS_d}$ of *discrete enhanced measurable sheaves* analogous to the Schanuel topos, and how separation-as-independent-combination naturally lives in a category $\mathbf{Set}^{\ll}_{\mathrm{d}}$ of *discrete absolutely continuous sets*. Finally, we show these two categories equivalent, and that the two notions of separation correspond across this equivalence, giving an analog of Fact 1.1 suitable for discrete probability.

In Section 2 we considered a tiny separation logic TinySep for integer-valued stores. Analogously, we consider here a logic for integer-valued random variables:

$$P, Q ::= X \sim \mu \mid \mathsf{True} \mid P * Q. \qquad \text{(TinyProbSep)}$$

The proposition $X \sim \mu$ asserts that the logical variable X stands for an integer-valued random variable with probability mass function $\mu : \mathbb{Z} \to [0, 1]$. As in TinySep, a proposition is well-formed in context Γ, written $\Gamma \vdash P$, if Γ contains the variables used by P. We shall establish the equivalence of two different models for TinyProbSep. In both cases, the basic idea is that a proposition denotes a predicate on *probability spaces* and logical variables denote *random variables*, just as a proposition in ordinary separation logic denotes a predicate on stores with logical variables denoting heap locations. The difference is in how these objects are represented:

Model 1: separation as product. In this model, a *probability space* consists of two components: (1) a nonempty countable set Ω called the *sample space*, and (2) a *probability space* $\mathcal{P}$ *on* Ω consisting of a pair $(\mathcal{F}, \mu)$ with $\mathcal{F}$ a σ-algebra on Ω and $\mu : \mathcal{F} \to [0, 1]$ a probability measure. A *random variable on* Ω is a function $\Omega \to \mathbb{Z}$. We will write $\mathbb{P}(\Omega)$ and $\mathrm{RV}(\Omega)$ for the set of probability spaces and random variables on Ω respectively.

The meaning of a proposition depends on the underlying sample space: $\Gamma \vdash P$ denotes a map $[\![\Gamma \vdash P]\!]_1^\Omega : (\Gamma \to \mathrm{RV}(\Omega)) \to \mathscr{P}(\mathbb{P}(\Omega))$ associating each *random substitution* $G : \Gamma \to \mathrm{RV}(\Omega)$ to the set $[\![\Gamma \vdash P]\!]_1^\Omega(G)$ of probability spaces on Ω satisfying P.

Under this interpretation, we have $\mathcal{P} \in [\![\mathsf{True}]\!]_1^\Omega(G)$ for all probability spaces $\mathcal{P}$ on Ω, and $(\mathcal{F}, \mu) \in [\![X \sim \nu]\!]_1^\Omega(G)$ if and only if $G(X)$ is $\mathcal{F}$-measurable and has distribution ν; i.e., for all $i \in \mathbb{Z}$ it holds that $G(X)^{-1}(i) \in \mathcal{F}$ and $\mu(G(X)^{-1}(i)) = \nu(i)$. Separating conjunction is defined in terms of products of sample spaces. To make this precise, we need the following definitions:

Definition 3.1 (Pullback probability space)**.** Let X be a nonempty countable set, $(Y, \mathcal{G}, \nu)$ a countable probability space, and $f : X \twoheadrightarrow Y$ a surjective function. The *pullback of* $(\mathcal{G}, \nu)$ *along* f, written $f^{-1}(\mathcal{G}, \nu)$, is the probability space $(\mathcal{F}, \mu)$ on X defined by $\mathcal{F} = \{f^{-1}(G) \mid G \in \mathcal{G}\}$ and $\mu(f^{-1}(G)) = \nu(G)$. Note μ is well-defined because f surjective, so f^{-1} injective.

Definition 3.2 (Subspace)**.** Given two probability spaces $(\mathcal{F}, \mu)$ and $(\mathcal{G}, \nu)$ on Ω, say $(\mathcal{F}, \mu)$ is a *subspace* of $(\mathcal{G}, \nu)$, written $(\mathcal{F}, \mu) \sqsubseteq (\mathcal{G}, \nu)$, if $\mathcal{F} \subseteq \mathcal{G}$ and $\nu|_{\mathcal{F}} = \mu$.

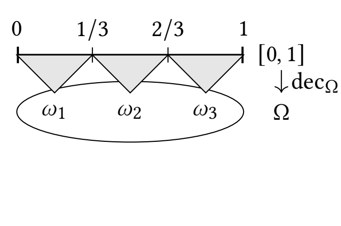
With these definitions, the separating conjunction $P_1 * P_2$ holds of a probability space $\mathcal{P}$ on Ω if and only if there exist probability spaces $\mathcal{P}_1$ on Ω_1 and $\mathcal{P}_2$ on Ω_2 and a surjective function $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$ such that $\mathcal{P}_1$ satisfies P_1 and $\mathcal{P}_2$ satisfies P_2 and $p^{-1}(\mathcal{P}_1 \otimes \mathcal{P}_2)$ is a subspace of $\mathcal{P}$, where $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the product probability space on $\Omega_1 \times \Omega_2$ whose measure is the product measure induced by the measures of $\mathcal{P}_1$ on Ω_1 and $\mathcal{P}_2$ on Ω_2 in the usual way.

For example, let Ω be the sample space $\{0, 1\}^3$ of points $(x, y, z) \in \mathbb{R}^3$ with x, y, z all either 0 or 1. Let G be the random substitution of type $\{X, Y\} \to \mathrm{RV}(\Omega)$ where $G(X)$ is the random variable $(x, y, z) \mapsto x$ and $G(Y)$ is the random variable $(x, y, z) \mapsto y$. Let $(\mathcal{F}, \mu)$ be the uniform probability space on Ω, assigning each tuple (x, y, z) probability $1/8$. It holds that

$$(\mathcal{F}, \mu) \in [\![(X \sim \mathrm{Ber}(1/2)) * (Y \sim \mathrm{Ber}(1/2))]\!]_1^\Omega(G),$$

witnessed by setting p to the projection $\Omega \twoheadrightarrow \{0, 1\} \times \{0, 1\}$ defined by $p(x, y, z) = (x, y)$.

Model 2: separation as independent combination. In this model, one fixes upfront a single measurable space to serve as a "universal sample space" into which all discrete sample spaces can be embedded. Any standard Borel space will do; we choose the interval $[0, 1]$. The idea is that, just as every finite store shape L can be encoded as a finite subset of $\mathbb{N}$ via an injective function $\mathrm{enc}_L : L \hookrightarrow \mathbb{N}$, every nonempty countable sample space Ω can be encoded as a countable partition of the interval via a random variable $\mathrm{dec}_\Omega : [0, 1] \to \Omega$ with each $\mathrm{dec}_\Omega^{-1}(\omega)$ nonnegligible, visualized as:



(IntervalEncode)

This illustration gives one possible encoding of the three-point space $\{\omega_1, \omega_2, \omega_3\}$ as the partition $\{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$, generated by the random variable $\mathrm{dec}_\Omega : [0, 1] \to \{\omega_1, \omega_2, \omega_3\}$ taking value ω_1 on $[0, 1/3)$, ω_2 on $[1/3, 2/3]$, and ω_3 on $(2/3, 1]$.

With this fixed sample space in hand, the random variables $\Omega \to \mathbb{Z}$ of Model 1 can be encoded as Lebesgue-measurable functions $[0, 1] \to \mathbb{Z}$, quotiented by almost-everywhere equality. We will write $\overline{\mathrm{RV}}$ for the set of integer-valued random variables.

In order for our encoding of sample spaces as partitions generated by random variables to respect almost-everywhere equality of random variables, we must consider such partitions up to negligibility: for example, the partitions $\{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$ and $\{[0, 1/3], (1/3, 2/3), [2/3, 1]\}$ should be considered equivalent, as they arise from almost-everywhere-equal random variables. This idea motivates the following definition.

Definition 3.3. A *countable measurable partition of* $[0, 1]$ is a countable partition $\{A_i\}_{i\in I}$ with each A_i a Lebesgue-measurable and non-negligible subset of $[0, 1]$, quotiented by almost-everywhere equality: two partitions are almost-everywhere equal, written $\{A_i\}_{i\in I} =_{\text{a.s.}} \{B_j\}_{j\in J}$, if for all i in I there exists a unique j in J such that the symmetric difference $A_i \triangle B_j$ is Lebesgue-negligible.

Just as any L-shaped valuation can be encoded as a finite partial function on $\mathbb{N}$, any discrete probability space can be encoded as a countable measurable partition equipped with a measure:

Definition 3.4. A *countable measured partition of* $[0, 1]$ is a pair $(\{A_i\}_{i\in I}, \mu)$ with $\{A_i\}_{i\in I}$ a countable measurable partition of $[0, 1]$ and $\mu : \{A_i\}_{i\in I} \to [0, 1]$ a function satisfying $\sum_i \mu(A_i) = 1$. Two such partitions are equal if their measurable partitions are equal and their measures agree. Let $\overline{\mathbb{P}}$ be the set of countable measured partitions of $[0, 1]$.

Now that we have a way of encoding discrete probability spaces as countable measured partitions of $[0, 1]$, we can define a model of TinyProbSep purely in terms of countable measured partitions. A proposition $\Gamma \vdash P$ denotes a map $[\![\Gamma \vdash P]\!]_2 : (\Gamma \to \overline{\mathrm{RV}}) \to \mathcal{P}(\overline{\mathbb{P}})$ assigning each random substitution $G : \Gamma \to \overline{\mathrm{RV}}$ the set of countable measured partitions satisfying P. The interpretations of True and $X \sim \mu$ are as in Model 1: $\mathcal{P} \in [\![\mathsf{True}]\!]_2\,(G)$ for any countable measured partition $\mathcal{P}$, and $(\{A_i\}_{i\in I}, \mu) \in [\![X \sim \nu]\!]_2\,(G)$ if and only if for all $k \in \mathbb{Z}$ there exists $i \in I$ with $G(X)^{-1}(k) \triangle A_i$ negligible and $\mu(G(X)^{-1}(k)) = \nu(k)$. Separating conjunction is defined via *independent combination*, following Li et al. [29]:

Definition 3.5 (Discrete independent combination)**.** A countable measured partition $(\mathcal{A}, \mu)$ is an *independent combination* of $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$ if (1) the partition $\mathcal{A}$ is generated by the intersections $A_1 \cap A_2$ for A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$ and (2) $\mu(A_1 \cap A_2) = \mu_1(A_1)\mu_2(A_2)$ for all A_1 in $\mathcal{A}_1$ and A_2 in $\mathcal{A}_2$. Independent combinations are unique if they exist [29, Lemma 2.3], defining a partial function $\bar{\bullet}$ with $(\mathcal{A}_1, \mu_1) \mathbin{\bar{\bullet}} (\mathcal{A}_2, \mu_2) = (\mathcal{A}, \mu)$ if and only if $(\mathcal{A}, \mu)$ is the independent combination of $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$.

Definition 3.6 (Ordering on partitions)**.** For two countable measured partitions $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$, let $\overline{\sqsubseteq}$ be the ordering relation defined by $(\mathcal{A}, \mu) \mathrel{\overline{\sqsubseteq}} (\mathcal{B}, \nu)$ if and only if the partition $\mathcal{A}$ is coarser than $\mathcal{B}$ and ν restricts to μ.

These definitions give an interpretation of separating conjunction: a countable measured partition $(\mathcal{A}, \mu)$ on $[0, 1]$ satisfies $P_1 * P_2$ with random substitution G, written $(\mathcal{A}, \mu) \in [\![P_1 * P_2]\!]_2\,(G)$, if and only if there exist $(\mathcal{A}_1, \mu_1)$ and $(\mathcal{A}_2, \mu_2)$ independently combinable with $(\mathcal{A}_1, \mu_1) \mathbin{\bar{\bullet}} (\mathcal{A}_2, \mu_2) \mathrel{\overline{\sqsubseteq}} (\mathcal{A}, \mu)$ such that $(\mathcal{A}_1, \mu_1)$ is in $[\![P_1]\!]_2\,(G)$ and $(\mathcal{A}_2, \mu_2)$ is in $[\![P_2]\!]_2\,(G)$.

Relating the two models. We will show that Model 1 and Model 2 are equivalent. As shown in IntervalEncode, every nonempty countable sample space Ω can be encoded as a countable measured partition on $[0, 1]$ via a suitable random variable $\mathrm{dec}_\Omega : [0, 1] \to \Omega$. Choosing a dec_Ω for every Ω allows translating Model 1 into Model 2: a Model 1 random variable $X \in \mathrm{RV}(\Omega)$ corresponds to a Model 2 random variable $X \circ \mathrm{dec}_\Omega \in \overline{\mathrm{RV}}$, and probability spaces can be translated similarly. To extend this into an equivalence analogous to Fact 1.1, we repeat the recipe of Section 2: we will place Models 1 and 2 into suitable categories, show the categories equivalent, and show that the models correspond across this equivalence.

3.1 Discrete enhanced measurable sheaves

In Section 2.1 we saw how the Schanuel topos **Sch** captured the invariants maintained by Model 1 of TinySep. In this section we describe analogously how a category of *discrete enhanced measurable sheaves*, written $\mathbf{EMS}_\mathrm{d}$, captures the invariants maintained by Model 1 of TinyProbSep. Whereas the invariants of Section 2.1 were about extensions and restrictions of the store shape L, the invariants in our probabilistic setting are about extensions and restrictions of the sample space Ω, as observed by Simpson [46, 47]:

- *Extension*: propositions that hold in sample space Ω should continue to hold when Ω is extended to a larger sample space Ω' via a surjective function $p : \Omega' \twoheadrightarrow \Omega$. More precisely, if $(\mathcal{F}, \mu) \in [\![\Gamma \vdash P]\!]_1^{\Omega}\,(G)$ for some probability space $(\mathcal{F}, \mu)$ on Ω and random substitution $G : \Gamma \to \mathrm{RV}(\Omega)$, then it should hold that $p^{-1}(\mathcal{F}, \mu) \in [\![\Gamma \vdash P]\!]_1^{\Omega'}\,(G \cdot p)$, where $G \cdot p$ is the random substitution $(G \cdot p)(X) = G(X) \circ p$.[5]
- *Restriction*: the truth of a proposition should not depend on any unused samples. For example, let Ω be an arbitrary sample space. Suppose $G : \Gamma \to \mathrm{RV}(\Omega)$ sends every X in Γ to the constant random variable 0, so $G(X)(\omega) = 0$ for all ω, and let $(\mathcal{F}, \mu)$ be the minimal probability space on Ω where $\mathcal{F}$ is the minimal σ-algebra $\{\emptyset, \Omega\}$ and μ the minimal probability measure with $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Both G and $(\mathcal{F}, \mu)$ don't use any of the samples in Ω: every random variable $G(X)$ is a deterministic value, and μ only assigns probabilities to the deterministic events $\emptyset$ and Ω. Restriction says that if a proposition P holds in this situation, then it should hold of the one-point probability space on the one-point set with substitution sending every X in Γ to the constant random variable 0.

To capture these invariants, we replay the construction of the Schanuel topos: whereas the Schanuel topos is a category of atomic sheaves on the category **Shp** of store shapes, $\mathbf{EMS}_\mathrm{d}$ is a category of atomic sheaves on a category of discrete sample spaces.

[5] Note that this rolls the two invariants Extension and Renaming of Section 2.1 into one: it captures invariance under permutations of the underlying sample space in the case where p is a bijection.

First, we fix a base category capturing Extension: the category $\mathbf{Surj}_{\leq\omega}$ of nonempty countable sets and surjective functions. The idea is that an object Ω of $\mathbf{Surj}_{\leq\omega}$ is a countable sample space, and a morphism $p : \Omega' \twoheadrightarrow \Omega$ extends a sample space Ω to a larger space Ω' in which every sample ω in Ω is expanded to a set of samples $p^{-1}(\omega) \subseteq \Omega'$; surjectivity of p ensures that every $p^{-1}(\omega)$ is nonempty, so p never deletes any samples in Ω.

Functors $\mathbf{Surj}_{\leq\omega}^{\mathrm{op}} \to \mathbf{Set}$ model sample-space-dependent concepts. In particular, there is a functor modelling probability spaces: For a $\mathbf{Surj}_{\leq\omega}$-morphism $p : \Omega' \twoheadrightarrow \Omega$, setting $\mathbb{P}(p)$ to the function $\mathbb{P}(\Omega) \to \mathbb{P}(\Omega')$ that sends a probability space $\mathcal{P}$ on Ω to its pullback $p^{-1}\mathcal{P}$ makes $\mathbb{P}$ a functor $\mathbf{Surj}_{\leq\omega}^{\mathrm{op}} \to \mathbf{Set}$.

Next, we capture Restriction by cutting the functor category $[\mathbf{Surj}_{\leq\omega}^{\mathrm{op}}; \mathbf{Set}]$ down to a full subcategory of atomic sheaves. The notion of atomic sheaf is given by the notion of *atomic topology*, which exists for a given category if and only if the following *Ore property* holds [30, p.115]:

Definition 3.7. A category C has the *right Ore property* if for all $X \xrightarrow{f} Z \xleftarrow{g} Y$ there exists $X \xleftarrow{h} W \xrightarrow{k} Y$ such that $fh = gk$.

That $\mathbf{Surj}_{\leq\omega}$ satisfies this condition can be straightforwardly verified: any cospan can be completed to a commutative square by taking a pullback in $\mathbf{Set}$. Thus the notion of atomic sheaf makes sense for $\mathbf{Surj}_{\leq\omega}$, a functor is an atomic sheaf if and only if it has a restriction operation in the sense of Definition 2.6, and the following definition makes sense:

Definition 3.8. Let $\mathbf{EMS}_{\mathrm{d}}$ be the full subcategory of the category $[\mathbf{Surj}_{\leq\omega}^{\mathrm{op}}; \mathbf{Set}]$ consisting of those functors that are atomic sheaves.

Just as $\mathbb{P}(\Omega)$ models the concept of probability spaces on Ω, there are other atomic sheaves corresponding to each of the other concepts used to define Model 1:

Proposition 3.9. The following are objects of $\mathbf{EMS}_{\mathrm{d}}$:

- The constant functor Prop of propositions sending every object of $\mathbf{Surj}_{\leq\omega}$ to the set $\{\top, \bot\}$ and every morphism of $\mathbf{Surj}_{\leq\omega}$ to the identity function.
- The functor RV of random variables, with action on morphisms defined by $\mathrm{RV}(p : \Omega' \twoheadrightarrow \Omega)(X : \mathrm{RV}(\Omega)) = (X \circ p : \mathrm{RV}(\Omega'))$.
- The functor RV^Γ of Γ-substitutions with $\mathrm{RV}^\Gamma(\Omega) = \Gamma \to \mathrm{RV}(\Omega)$ and action on morphisms defined by lifting RV pointwise.

With these sheaves in hand, one can show Model 1 lives in $\mathbf{EMS}_{\mathrm{d}}$:

Proposition 3.10. If $\Gamma \vdash P$ then the Ω-indexed family of functions

$$\left(\llbracket \Gamma \vdash P \rrbracket_1^\Omega : (\Gamma \to \mathrm{RV}(\Omega)) \to \mathcal{P}(\mathbb{P}(\Omega))\right)_{\Omega \in \mathbf{Surj}_{\leq\omega}}$$

is natural in Ω, so defines a morphism $\mathrm{RV}^\Gamma \to \mathrm{Prop}^{\mathbb{P}}$ in $\mathbf{EMS}_{\mathrm{d}}$, and every morphism of this type satisfies Extension and Restriction.

3.2 Discrete absolutely continuous sets

We now turn to Model 2 of TinyProbSep described in Section 3. Just as Model 2 of TinySep naturally lives in the category $\mathbf{Nom}$ of nominal sets, Model 2 of TinyProbSep naturally lives in a category $\mathbf{Set}_{\mathrm{d}}^{\ll}$ of *discrete absolutely continuous sets*. $\mathbf{Nom}$ captures two invariants held by Model 2 of TinySep: Permutation and Finiteness. Model 2 of TinyProbSep maintains analogous invariants:

- *Permutation*: propositions should be stable under permuting the sample space $[0, 1]$. More precisely, if $(\mathcal{A}, \mu) \in \llbracket \Gamma \vdash P \rrbracket_2 (G)$ for some countable measured partition $(\mathcal{A}, \mu)$ and random substitution $G : \Gamma \to \overline{\mathrm{RV}}$, and if $\pi : [0, 1] \to [0, 1]$ is a measurable bijection, then it should hold that $(\mathcal{A}, \mu) \cdot \pi \in \llbracket \Gamma \vdash P \rrbracket_2 (G \cdot \pi)$, where $(\mathcal{A}, \mu) \cdot \pi$ and $G \cdot \pi$ are the results of the permutation π acting on $(\mathcal{A}, \mu)$ and G.
- *Sparsity*: more subtly, the countable measured partitions $(\mathcal{A}, \mu)$ represent *countable* probability spaces only. This ensures $(\mathcal{A}, \mu)$ always leaves "enough room" in $[0, 1]$ for "fresh randomness": for any other discrete probability space, there exists an encoding of it as a countable measured partition $(\mathcal{B}, \nu)$ such that the discrete independent combination $(\mathcal{A}, \mu) \mathbin{\bar{\bullet}} (\mathcal{B}, \nu)$ is defined.

To capture Permutation, the objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$ are sets equipped with an action by a group of measurable automorphisms. Specifically, let $\mathrm{Aut}[0, 1]$ be the group of measurable maps $\pi : [0, 1] \to [0, 1]$ that are bijective mod almost-everywhere equality. The category of $\mathrm{Aut}[0, 1]$-sets is the category whose objects are sets X equipped with a right action by $\mathrm{Aut}[0, 1]$ and whose morphisms are equivariant functions. Just as there is a S_ω-set $\overline{\mathrm{S}}$ of stores, there is a $\mathrm{Aut}[0, 1]$-set $\overline{\mathbb{P}}$ of countable measured partitions on $[0, 1]$:

Definition 3.11. Let $\overline{\mathbb{P}}$ be the set of countable measured partitions on $[0, 1]$ with action $(\{A_i\}_{i \in I}, \mu) \cdot \pi = (\{\pi^{-1}(A_i)\}_{i \in I}, \mu \circ \pi)$.

Sparsity is captured by topologizing $\mathrm{Aut}[0, 1]$ via countable measurable partitions, so a stabilizer $\mathrm{Stab}\, x$ is open if every π stabilizing x does so for a "countable reason": there is a partition $\mathcal{A}$ fixed by π such that any other permutation fixing $\mathcal{A}$ also stabilizes x.

Definition 3.12 (Topology on $\mathrm{Aut}[0, 1]$)**.** A subset U of $\mathrm{Aut}[0, 1]$ is *open* if for every $\pi \in U$, there exists a countable measurable partition $\mathcal{A}$ of $[0, 1]$ such that $\pi \in \mathrm{Fix}\, \mathcal{A} \subseteq U$, where $\mathrm{Fix}\, \mathcal{A}$ is the subgroup of $\mathrm{Aut}[0, 1]$ consisting of those permutations π that fix every element of $\mathcal{A}$; i.e., $\pi(A) =_{\mathrm{a.e.}} A$ for all $A \in \mathcal{A}$.

Definition 3.13. A *discrete absolutely continuous set* is a $\mathrm{Aut}[0, 1]$-set whose elements have open stabilizers. Let $\mathbf{Set}_{\mathrm{d}}^{\ll}$ be the category of discrete absolutely continuous sets and equivariant functions.

In addition to $\mathbb{P}$, there are objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$ corresponding to each of the other concepts used to define Model 2 of TinyProbSep:

Proposition 3.14. The following are objects of $\mathbf{Set}_{\mathrm{d}}^{\ll}$:

- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{Prop}} = \{\top, \bot\}$ with the trivial action.
- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{RV}}$ of random variables with action $X \cdot \pi = X \circ \pi$.
- The $\mathrm{Aut}[0, 1]$-set $\overline{\mathrm{RV}}^\Gamma$ of random Γ-substitutions $\Gamma \to \overline{\mathrm{RV}}$ with action defined by lifting $\overline{\mathrm{RV}}$ pointwise.

With these in hand, one can show Model 2 lives in $\mathbf{Set}_{\mathrm{d}}^{\ll}$:

Proposition 3.15. If $\Gamma \vdash P$ then the function $\llbracket \Gamma \vdash P \rrbracket_2$ is a morphism $\overline{\mathrm{RV}}^\Gamma \to \overline{\mathrm{Prop}}^{\overline{\mathbb{P}}}$ in $\mathbf{Set}_{\mathrm{d}}^{\ll}$, and every morphism of this type satisfies Permutation and Finiteness.

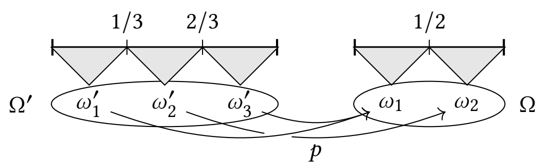
3.3 The equivalence of categories

Having placed Models 1 and 2 of TinyProbSep described in Section 3 into the categories $\mathbf{EMS}_{\mathrm{d}}$ and $\mathbf{Set}_{\mathrm{d}}^{\ll}$ respectively, we describe in this section how $\mathbf{EMS}_{\mathrm{d}}$ are $\mathbf{Set}_{\mathrm{d}}^{\ll}$ equivalent, giving an analog of **Sch** $\simeq$ **Nom** for discrete probability. The key step is to establish probabilistic analogs of Homogeneity and Correspondence:

Lemma 3.16 (Homogeneity)**.** Let Ω, Ω' be nonempty countable sets and let dec_Ω and $\mathrm{dec}_{\Omega'}$ be measurable functions $[0,1] \to \Omega$ and $[0,1] \to \Omega'$ with $\mathrm{dec}_\Omega^{-1}(\omega)$ and $\mathrm{dec}_{\Omega'}^{-1}(\omega')$ nonnegligible for all $\omega \in \Omega$ and $\omega' \in \Omega'$. For any surjective function $f : \Omega' \twoheadrightarrow \Omega$, there exists $\pi \in \mathrm{Aut}[0,1]$ making the following square commute:

$$\begin{tikzcd} {[0,1]} \arrow[r, dashed, "\pi"] \arrow[d, "\mathrm{dec}_{\Omega'}"'] & {[0,1]} \arrow[d, "\mathrm{dec}_\Omega"] \\ \Omega' \arrow[r, two heads, "p"'] & \Omega \end{tikzcd}$$

This lemma is particularly important, so we give some intuition about its proof. Consider the following visualization of a surjection p from $\Omega' = \{\omega_1', \omega_2', \omega_3'\}$ onto $\Omega = \{\omega_1, \omega_2\}$ and two decoding functions $\mathrm{dec}_{\Omega'}$ and dec_Ω visualized as in IntervalEncode:



Lemma 3.16 asserts that there exists π with $p \circ \mathrm{dec}_{\Omega'} = \mathrm{dec}_\Omega \circ \pi$. Indeed we can explicitly construct such a π for this example: let π be an automorphism that sends the interval $[0, 1/3]$ to $[0, 1/4]$, the interval $[2/3, 1]$ to $[1/4, 1/2]$, and finally $[1/3, 2/3]$ to $[1/2, 1]$. This construction generalizes nicely to any situation where the preimages $\mathrm{dec}_{\Omega'}^{-1}(\omega_i')$ and $\mathrm{dec}_\Omega^{-1}(\omega_i)$ are all intervals. In the fully general case, these preimages can be arbitrary Lebesgue-measurable sets, but every such set is measurably isomorphic to an interval [19, 344J], so the general case reduces to the one sketched above.

Lemma 3.17 (Correspondence)**.** For any countable measurable partition $\{A_i\}_{i\in I}$ of $[0,1]$, let $\mathrm{Fix}\{A_i\}_{i\in I}$ be the subgroup of $\mathrm{Aut}[0,1]$ consisting of those automorphisms $\pi \in \mathrm{Aut}[0,1]$ fixing $\{A_i\}_{i\in I}$, so that $\pi(A_i) \triangle A_i$ negligible for all $i \in I$. For any two partitions $\mathcal{A}$ and $\mathcal{B}$, it holds that $\mathrm{Fix}\,\mathcal{A} \subseteq \mathrm{Fix}\,\mathcal{B}$ iff $\mathcal{A}$ is finer than $\mathcal{B}$.

Proof. If $\mathcal{A}$ is finer than $\mathcal{B}$ then certainly every π fixing $\mathcal{A}$ fixes $\mathcal{B}$. For the converse, suppose for contradiction that $\mathcal{A}$ is not finer than $\mathcal{B}$, so there is some $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ with $A \cap B_1$ and $A \cap B_2$ both nonnegligible. Pick an arbitrary π swapping $A \cap B_1$ with $A \cap B_2$; π fixes $\mathcal{A}$ but not $\mathcal{B}$, contradicting $\mathrm{Fix}\,\mathcal{A} \subseteq \mathrm{Fix}\,\mathcal{B}$. ◻

The equivalence follows from these lemmas:

Theorem 3.18. $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

For details, see the appendix: Theorem 3.18 follows from a specialization of Theorem C.33 using Lemmas 3.16 and 3.17 to satisfy the preconditions. This equivalence of categories extends to an equivalence of Models 1 and 2 of TinyProbSep. The argument is as in Section 2.3: we package Models 1 and 2 into resource monoids in $\mathbf{EMS}_\mathrm{d}$ and $\mathbf{Set}_\mathrm{d}^{\ll}$ respectively, and then show they correspond across the equivalence $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

To construct the resource monoid packaging Model 1 into $\mathbf{EMS}_\mathrm{d}$, we make use of a general recipe for constructing models of separation logic via the *Day convolution* [7, 15, 35]. The Day convolution is a general construction lifting a monoidal structure on a base category C to a monoidal structure on $[C^{\mathrm{op}}; \mathbf{Set}]$, see Day [14]. The resource monoid $(\sqsubseteq, \mathrm{S}^2_\perp, i, \bullet)$ in **Sch** described in Section 2.3 can be constructed using the Day convolution: the base category **Shp** has a monoidal product given by coproduct of finite sets, and Day convolution lifts this to a monoidal product $\otimes$ on $[\mathbf{Shp}^{\mathrm{op}}; \mathbf{Set}]$; applying the Day convolution to the sheaf S gives the functor $\mathrm{S} \otimes \mathrm{S}$, which one can show is naturally isomorphic to $\mathrm{S}^2_\perp$; the operations $\sqsubseteq, i, \bullet$ can then be defined straightforwardly.

To apply this recipe for discrete probability, we replace **Shp** with $\mathbf{Surj}_{\leq\omega}$ and coproduct $+$ of finite sets with product $\times$ of sample spaces. This makes $(\mathbf{Surj}_{\leq\omega}, \times, 1)$ a monoidal category, where the unit 1 is the one-point sample space. Via the Day convolution, $\times$ lifts to a monoidal product $\otimes$ on $[\mathbf{Surj}_{\leq\omega}^{\mathrm{op}}; \mathbf{Set}]$. Just as $\mathrm{S} \otimes \mathrm{S}$ is isomorphic to the functor $\mathrm{S}^2_\perp$ modelling separated stores, the Day convolution $\mathbb{P} \otimes \mathbb{P}$ is isomorphic to a sheaf of probability spaces that can be rendered independent with a suitable joint measure:

Proposition 3.19. The functor $\mathbb{P} \otimes \mathbb{P}$ is isomorphic to an atomic sheaf $\mathbb{P}^2_\perp$ sending each $\Omega \in \mathbf{Surj}_{\leq\omega}$ to the set

$$\{\, ((\pi_1 \circ p)^{-1}(\mathcal{P}_1), (\pi_2 \circ p)^{-1}(\mathcal{P}_2)) \\ \mid \Omega_1, \Omega_2 \in \mathbf{Surj}_{\leq\omega}, \mathcal{P}_1 \in \mathbb{P}(\Omega_1), \mathcal{P}_2 \in \mathbb{P}(\Omega_2), p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2 \,\}$$

of pairs of probability spaces on Ω that "factor through" a product $\Omega_1 \times \Omega_2$ along some projection $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$.

The resource monoid operations can be defined as follows. First, the subspace ordering $\sqsubseteq$ forms a natural transformation $(\sqsubseteq) : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$. Next, there is natural transformation $(\bullet) : \mathbb{P}^2_\perp \to \mathbb{P}$ sending a pair $((\pi_1 \circ p)^{-1}(\mathcal{P}_1), (\pi_2 \circ p)^{-1}(\mathcal{P}_2))$ of probability spaces that factor through some $p : \Omega \twoheadrightarrow \Omega_1 \times \Omega_2$ to $p^{-1}(\mathcal{P}_1 \otimes \mathcal{P}_2)$, where $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the usual product probability space on $\Omega_1 \times \Omega_2$. Each $\mathbb{P}^2_\perp(\Omega)$ is a subset of $(\mathbb{P} \times \mathbb{P})(\Omega)$, and collecting the canonical subset-inclusions into an Ω-indexed family forms a natural transformation $i : \mathbb{P}^2_\perp \hookrightarrow \mathbb{P} \times \mathbb{P}$. Finally, $\bullet$ is associative and commutative and monotone with respect to $\sqsubseteq$, and has unit the natural transformation $\mathrm{emp} : 1 \to \mathbb{P}$ sending a sample space Ω to the trivial probability space $(\Omega, \{\emptyset, \Omega\}, \mu)$ with $\mu(\Omega) = 1$.

Proposition 3.20. $(\sqsubseteq, \mathbb{P}^2_\perp, i, \bullet, \mathrm{emp})$ is a resource monoid in $\mathbf{EMS}_\mathrm{d}$.

Model 2 can be packaged into a resource monoid in $\mathbf{Set}_\mathrm{d}^{\ll}$ analogously. Let $\overline{\mathbb{P}}^2_\perp$ be the discrete absolutely continuous set $\{(\mathcal{P}_1, \mathcal{P}_2) \mid \mathcal{P}_1, \mathcal{P}_2 \in \overline{\mathbb{P}}, \mathcal{P}_1 \mathbin{\bar{\bullet}} \mathcal{P}_2 \text{ defined}\}$ of pairs of independently combinable countable measured partitions of $[0,1]$ with pointwise group action. This is a subset of $\overline{\mathbb{P}} \times \overline{\mathbb{P}}$; both the canonical inclusion map $\bar{i}$ and the function $\bar{\bullet} : \overline{\mathbb{P}}^2_\perp \to \overline{\mathbb{P}}$ sending a pair $(\mathcal{P}_1, \mathcal{P}_2)$ of independently combinable probability spaces on $[0,1]$ to their independent combination $\mathcal{P}_1 \mathbin{\bar{\bullet}} \mathcal{P}_2$ are equivariant. Finally, the ordering relation $\overline{\sqsubseteq}$ on probability spaces on $[0,1]$ is equivariant, so defines a morphism $(\overline{\sqsubseteq}) : \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \overline{\mathrm{Prop}}$, and this ordering relation has as least element $\overline{\mathrm{emp}}$ the measured partition containing a single component with probability 1.

The following theorem establishes that $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{i}, \bar{\bullet})$ is a resource monoid together with an analog of Fact 1.1 for discrete probability:

Theorem 3.21. The resource monoid $(\sqsubseteq, \mathbb{P}^2_\perp, i, \bullet, \mathrm{emp})$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ across the equivalence $\mathbf{EMS}_\mathrm{d} \simeq \mathbf{Set}_\mathrm{d}^{\ll}$.

4 THE CONTINUOUS CASE

In this section we generalize Section 3 from discrete to continuous probability: $\mathbf{EMS}_\mathrm{d}$ becomes a category $\mathbf{EMS}$ of *enhanced measurable sheaves*, and $\mathbf{Set}_\mathrm{d}^{\ll}$ becomes a category $\mathbf{Set}^{\ll}$ of *absolutely continuous sets*. Due to the amount of measure theory required, we stick to stating the key definitions and lemmas; the full details can be found in Appendices D and E.

4.1 Enhanced measurable sheaves

The first step in generalizing $\mathbf{EMS}_\mathrm{d}$ to $\mathbf{EMS}$ is to replace the base category $\mathbf{Surj}_{\leq\omega}$ of discrete sample spaces with a base category of continuous sample spaces.

The starting point for this generalization is the following observation. Let $\mathbf{Prob}^+_{\leq\omega}$ be the category whose objects are countable probability spaces $(\Omega, \mu : \Omega \to [0,1])$ with $\mu(\omega) > 0$ for all $\omega \in \Omega$, and whose morphisms $(\Omega,\mu) \to (\Omega',\mu')$ are measure-preserving maps $f : \Omega \to \Omega'$; i.e., $\sum_{f(\omega)=\omega'} \mu(\omega) = \mu'(\omega')$ for all $\omega' \in \Omega'$. There is a functor $\mathrm{U_d} : \mathbf{Prob}^+_{\leq\omega} \to \mathbf{Surj}_{\leq\omega}$ that forgets the measures μ: measure-preserving maps f between probability spaces with strictly positive measure are surjective because $f^{-1}(y)$ must be nonempty for all $y \in \mathrm{cod}(f)$. The category $\mathbf{Surj}_{\leq\omega}$ is the image of $\mathbf{Prob}^+_{\leq\omega}$ under $\mathrm{U_d}$: every nonempty countable set Ω can be equipped with a strictly positive probability measure, and for every surjective function $p : \Omega' \twoheadrightarrow \Omega$, there exist strictly positive probability measures μ' on Ω' and μ on Ω making p a measure-preserving map $(\Omega',\mu') \to (\Omega,\mu)$. Thus $\mathbf{Surj}_{\leq\omega}$ can be thought of as a category of probability spaces where one has forgotten all measures.

To generalize this situation from discrete to continuous probability, we replace the category $\mathbf{Prob}^+_{\leq\omega}$ of countable probability spaces with a category of continuous probability spaces:

Definition 4.1 (The category $\mathbf{Prob}_\mathrm{std}$)**.** Let $\mathbf{Prob}_\mathrm{std}$ be the category of *standard probability spaces* [42] and measure-preserving maps quotiented by almost-everywhere equality.

Then, we replace the functor $\mathrm{U_d} : \mathbf{Prob}^+_{\leq\omega} \to \mathbf{Surj}_{\leq\omega}$ with a functor U that forgets continuous probability measures. The idea behind this forgetting process is as follows. Given a probability space $(X,\mathcal{F},\mu)$, one can forget everything about the measure μ except for which subsets are negligible, leaving behind an *enhanced measurable space* $(X,\mathcal{F},\mathcal{N})$, where $\mathcal{N}$ is the σ-ideal of μ-negligible sets [36, Definition 4.4]. Given a measure-preserving map $[f] : (X,\mathcal{F},\mu) \to (Y,\mathcal{G},\nu)$ quotiented by almost-everywhere equality where μ has negligibles $\mathcal{N}$ and ν has negligibles $\mathcal{M}$, one can forget everything about $[f]$ measure-preserving except for the fact that $\nu(G) = 0$ iff $\mu(f^{-1}(G)) = 0$, leaving behind an equivalence class $[f]$ with $f^{-1}(G) \in \mathcal{N}$ iff $G \in \mathcal{M}$ for all $G \in \mathcal{G}$. This motivates the following definitions.

Definition 4.2 (The category $\mathbf{EMS}_\mathrm{std}$)**.** A *standard enhanced measurable space* is tuple $(X,\mathcal{F},\mathcal{N})$ for which there exists a measure μ making $(X,\mathcal{F},\mu)$ a standard probability space with negligibles $\mathcal{N}$. Given enhanced measurable spaces $(X,\mathcal{F},\mathcal{N})$ and $(Y,\mathcal{G},\mathcal{M})$, a measurable map $f : (X,\mathcal{F}) \to (Y,\mathcal{G})$ is *negligible-preserving and reflecting* if $f^{-1}(G) \in \mathcal{N}$ iff $G \in \mathcal{M}$ for all $G \in \mathcal{G}$; two such maps f, f' are *almost-everywhere equal* if $f^{-1}(G) \triangle f'^{-1}(G') \in \mathcal{N}$ for all $G, G' \in \mathcal{G}$ with $G \triangle G' \in \mathcal{M}$. Let $\mathbf{EMS}_\mathrm{std}$ be the category of standard enhanced measurable spaces and negligible-preserving-and-reflecting maps quotiented by almost-everywhere equality.

Proposition 4.3. Let $\mathrm{U} : \mathbf{Prob}_\mathrm{std} \to \mathbf{EMS}_\mathrm{std}$ be the functor that sends probability spaces $(X,\mathcal{F},\mu)$ with negligibles $\mathcal{N}$ to enhanced measurable spaces $(X,\mathcal{F},\mathcal{N})$. This functor is surjective on objects, and any morphism of standard enhanced measurable spaces arises from a measure-preserving map equipping those spaces with standard probability measures.

Then, just as $\mathbf{EMS}_\mathrm{d}$ is the category of atomic sheaves on $\mathbf{Surj}_{\leq\omega}$, $\mathbf{EMS}$ is the category of atomic sheaves on $\mathbf{EMS}_\mathrm{std}$:

Proposition 4.4. $\mathbf{EMS}_\mathrm{std}$ has the right Ore property.

Definition 4.5. Let $\mathbf{EMS}$ be the full subcategory of $[\mathbf{EMS}_\mathrm{std}^\mathrm{op}; \mathbf{Set}]$ consisting of atomic sheaves. Objects of $\mathbf{EMS}$ will be called *enhanced measurable sheaves*.

Inside $\mathbf{EMS}$, there are continuous analogs of the discrete enhanced measurable sheaves RV of random variables and $\mathbb{P}$ of discrete probability spaces. The continuous analog of RV models A-valued random variables for A Polish, following Simpson [46, 47]:

Definition 4.6. For any measurable space $(A,\mathcal{G})$ arising from a Polish space, the *sheaf of random variables* is:

$$\mathrm{RV}_A(\Omega,\mathcal{F},\mathcal{N}) = \{\text{measurable maps } (\Omega,\mathcal{F}) \to (A,\mathcal{G})\} \,/ =_{\mathcal{N}\text{-a.e.}}$$
$$\mathrm{RV}_A(p : \Omega' \to \Omega)([X] : \mathrm{RV}_A(\Omega)) : \mathrm{RV}_A(\Omega') = [X \circ p]$$

For proof that RV_A is indeed a sheaf, see Lemma D.3. Next, to generalize $\mathbb{P}$ from discrete to continuous probability, we make use of the following observation: every discrete probability space $(\Omega,\mathcal{F},\mu)$ arises via pullback from a surjection $X : \Omega \twoheadrightarrow A$ in which the set A is equipped with a probability mass function $\nu : A \to [0,1]$, by setting $\mathcal{F} := \{X^{-1}(a) \mid a \in A\}$ and $\mu(X^{-1}(a)) = \nu(a)$. Thus, discrete probability spaces $(\Omega,\mathcal{F},\mu)$ can be represented by $\mathbf{Surj}_{\leq\omega}$ morphisms $\Omega \to \mathrm{U_d}(A,\mu)$ for $(A,\mu) \in \mathbf{Prob}^+_{\leq\omega}$, where $\mathrm{U_d}$ is the functor $\mathbf{Surj}_{\leq\omega} \to \mathbf{Prob}^+_{\leq\omega}$ that forgets measures. This motivates the following generalization to the continuous setting.

Definition 4.7. The *sheaf of probability spaces* is

$$\mathbb{P} := \mathrm{colim}_{A : \mathrm{Core}(\mathbf{Prob}_\mathrm{std})} \, よ(\mathrm{U}A),$$

where $よ$ is the Yoneda embedding, $\mathrm{Core}(\mathbf{Prob}_\mathrm{std})$ is the subcategory of $\mathbf{Prob}_\mathrm{std}$-isomorphisms, and the colimit is taken in presheaves. (See Definition D.6 for proof that $\mathbb{P}$ is indeed a sheaf.) Concretely, the presheaf $\mathbb{P}$ sends $(\Omega,\mathcal{F},\mathcal{N}) : \mathbf{EMS}_\mathrm{std}$ to the set of pairs $((A,\mathcal{G},\mu), X)$ where $(A,\mathcal{G},\mu) : \mathbf{Prob}_\mathrm{std}$ and X is a $\mathbf{EMS}_\mathrm{std}$-map from $(\Omega,\mathcal{F},\mathcal{N})$ to $\mathrm{U}(A,\mathcal{G},\mu)$, quotiented by $((A,\mathcal{G},\mu),X) \sim ((A',\mathcal{G}',\mu'),X')$ iff there is a $\mathbf{Prob}_\mathrm{std}$-iso $i : (A,\mathcal{G},\mu) \to (A',\mathcal{G}',\mu')$ with $X' = U(i)\,X$. The action on morphisms is given by precomposition.

Using RV and $\mathbb{P}$, we generalize the resource monoid of Theorem 3.21 to a resource monoid of continuous probability spaces. The monoidal category $(\mathbf{Surj}_{\leq\omega}, \times, 1)$ of discrete sample spaces becomes a monoidal category $(\mathbf{EMS}_\mathrm{std}, \otimes, 1)$ of continuous sample spaces, with monoidal product $\otimes$ inherited from the usual tensor product $\otimes_{\mathbf{Prob}_\mathrm{std}}$ of standard probability spaces:

Definition 4.8. Given two standard enhanced measurable spaces X, Y, their *tensor product* $X \otimes Y$ is defined to be $\mathrm{U}(X' \otimes_{\mathbf{Prob}_\mathrm{std}} Y')$, where X' and Y' are arbitrary standard probability spaces with $\mathrm{U}(X') = X$ and $\mathrm{U}(Y') = Y$.

This is well-defined — the choice of X', Y' does not matter — and extends to a bifunctor on $\mathbf{EMS}_{\mathrm{std}}$ making $(\mathbf{EMS}_{\mathrm{std}}, \otimes, 1)$ a symmetric monoidal category with unit the one-point space 1. For details, see Appendix B.2. Lifting $\otimes$ to $[\mathbf{EMS}_{\mathrm{std}}^{\mathrm{op}}; \mathbf{Set}]$ via the Day convolution yields a resource monoid in $\mathbf{EMS}$:

Lemma 4.9. The Day convolution $\mathbb{P} \otimes \mathbb{P}$ is a sheaf, and there is a monic map of sheaves $i : \mathbb{P} \otimes \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$.

Lemma 4.10. There is a map of sheaves $\sqsubseteq : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$, where Prop is the constant sheaf at $\{\top, \bot\}$, and a map of sheaves $\mathrm{emp} : 1 \to \mathbb{P}$, making $(\mathbb{P}, \mathrm{emp})$ a poset in $\mathbf{EMS}$ with least element emp.

Lemma 4.11. There is a map of sheaves $\bullet : \mathbb{P} \otimes \mathbb{P} \to \mathbb{P}$, monotone with respect to $\sqsubseteq$, such that $(\mathbb{P}, \bullet, \mathrm{emp})$ is a partial commutative monoid in $\mathbf{EMS}$.

Theorem 4.12. $(\sqsubseteq, \mathbb{P}^2_\perp, i, \bullet, \mathrm{emp})$ is a resource monoid in $\mathbf{EMS}$.

For details, see Appendix D. While the colimit presentation of $\mathbb{P}$ makes it easier to check for sheafhood and to construct the above resource monoid, it is difficult to work with in the concrete calculations to follow. To address this, we show $\mathbb{P}$ equivalent to a sheaf of continuous probability spaces that arise via pullback along $\mathbf{EMS}_{\mathrm{std}}$-maps. To do this, we must take care to define pullback in a way that respects the negligible ideals contained in $\mathbf{EMS}_{\mathrm{std}}$-objects.

Definition 4.13. For $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ and $(Y, \mathcal{G}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$ and $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$, the *enhanced pullback of* $(Y, \mathcal{G}, \mu)$ *along* f, written $f^*(\mathcal{G}, \mu)$, is the pair $(f^*\mathcal{G}, f^*\mu)$ defined by

$$f^*\mathcal{G} = \{f^{-1}(G) \triangle N \mid G \in \mathcal{G}, N' \in \mathcal{N}\}$$
$$f^*\mu(f^{-1}(G) \triangle N) = \mu(G) \text{ for all } G \in \mathcal{G}, N \in \mathcal{N}$$

Enhanced pullback makes $(X, f^*\mathcal{G}, f^*\mu)$ a probability space with negligibles $\mathcal{N}$ and f a measure-preserving map $(X, f^*\mathcal{G}, f^*\mu) \to (Y, \mathcal{G}, \mu)$.

Definition 4.14. A *probability space on* $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ is a pair $(\mathcal{G}, \mu)$ with $\mathcal{N} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and μ a probability measure with negligibles $\mathcal{N}$. Call such a pair *standardizable* if $(X, \mathcal{G}, \mu)$ arises via enhanced pullback along a map $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$ for some $(Y, \mathcal{G}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$.

With these definitions in hand, the colimit $\mathbb{P}$ is equivalent to a sheaf of standardizable probability spaces, with action on morphisms given by enhanced pullback:

Lemma 4.15. $\mathbb{P}$ is equivalent to the following sheaf:

$$\hat{\mathbb{P}}(\Omega) = \{(\mathcal{G}, \mu) \mid (\mathcal{G}, \mu) \text{ standardizable on } \Omega\}$$
$$\hat{\mathbb{P}}(f : \Omega' \to \Omega)(\mathcal{G}, \mu) = f^*(\mathcal{G}, \mu)$$

Moreover, the Day convolution $\mathbb{P} \otimes \mathbb{P}$ corresponds to a sheaf of independently combinable probability spaces:

Lemma 4.16. $\mathbb{P} \otimes \mathbb{P}$ is equivalent to the following sheaf $\mathbb{P}^2_\perp$:

$$\mathbb{P}^2_\perp(\Omega) = \left\{ ((\mathcal{G}, \mu), (\mathcal{H}, \nu)) \;\middle|\; \begin{array}{l} (\mathcal{G}, \mu) \text{ and } (\mathcal{H}, \nu) \text{ standardizable} \\ \text{and independently combinable} \end{array} \right\}$$
$$\mathbb{P}^2_\perp(f : \Omega' \to \Omega)((\mathcal{G}, \mu), (\mathcal{H}, \nu)) = (f^*(\mathcal{G}, \mu), f^*(\mathcal{H}, \nu))$$

Via these equivalences, the resource monoid in Theorem 4.12 parallels its discrete analog (Proposition 3.20). Across $\mathbb{P} \cong \hat{\mathbb{P}}$, the ordering $\sqsubseteq$ corresponds to the generalization of Definition 3.6 from countable measured partitions to standardizable probability spaces. Across $\mathbb{P} \otimes \mathbb{P} \cong \mathbb{P}^2_\perp$, the monic map i corresponds to the canonical inclusion $\mathbb{P}^2_\perp \hookrightarrow \mathbb{P} \times \mathbb{P}$, and the combining operation $\bullet$ corresponds to the map $\mathbb{P}^2_\perp \to \mathbb{P}$ that sends independently-combinable pairs of standardizable probability spaces to their independent combination. For details, see Appendix E.2.

4.2 Absolutely continuous sets

Finding a continuous analog to $\mathbf{Set}^{\ll}_{\mathrm{d}}$ boils down to showing continuous analogs of Lemmas 3.16 and 3.17. In the discrete setting, these lemmas hold because every discrete probability space can be encoded as a measured partition that leaves enough room in the sample space $[0,1]$ for fresh randomness. To create a continuous analog, we fix an enormous sample space following Li et al. [29]:

Definition 4.17. The Hilbert cube $\mathbb{I}^\omega$ is the standard enhanced measurable space $([0,1]^\omega, \mathcal{F}, \mathcal{N})$ of infinite sequences in the interval $[0,1]$. The σ-algebra $\mathcal{F}$ and negligibles $\mathcal{N}$ are those of the usual Lebesgue measure on $[0,1]^\omega$.

Then, to ensure that there is always enough room left over in $\mathbb{I}^\omega$ for fresh randomness, we encode all probability spaces using only finitely many dimensions at a time:

Definition 4.18. A standardizable probability space $(\mathcal{G}, \mu)$ on $\mathbb{I}^\omega$ has *finite footprint* if it arises by enhanced pullback along a map $\mathbb{I}^\omega \to X$ that factors through $\mathrm{proj}_{1..n}$ for some n, where $\mathrm{proj}_{1..n}$ is the canonical projection $\mathbb{I}^\omega \to [0,1]^n$.

Analogously, the group $\mathrm{Aut}[0,1]$ of $\mathbf{Set}^{\ll}_{\mathrm{d}}$ becomes a group of finite-dimensional permutations of the Hilbert cube:

Definition 4.19. A $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi : \mathbb{I}^\omega \to \mathbb{I}^\omega$ has *finite width* if it is of the form $\pi' \times 1_{\mathbb{I}^\omega}$ for some $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi' : [0,1]^n \to [0,1]^n$. Let $\mathbf{G}^{\ll}$ be the subgroup of $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}} \mathbb{I}^\omega$ consisting of only those automorphisms with finite width.

Then, the topology on $\mathrm{Aut}[0,1]$ generated by countable measurable partitions becomes a topology on $\mathbf{G}^{\ll}$ generated by standardizable sub-σ-algebras with finite footprint:

Definition 4.20 (Topology on $\mathbf{G}^{\ll}$)**.** A subgroup U of $\mathbf{G}^{\ll}$ is *open* if for every π in U there exists $(\mathcal{F}, \mu)$ with finite footprint such that $\pi \in \mathrm{Fix}\,\mathcal{F} \subseteq U$, where $\mathrm{Fix}\,\mathcal{F}$ is the subgroup of those π in $\mathbf{G}^{\ll}$ with $\pi(F) =_{\mathrm{a.e.}} F$ for all $F \in \mathcal{F}$.

Definition 4.21. $\mathbf{Set}^{\ll}$ is the category of $\mathbf{G}^{\ll}$-sets with open stabilizers and equivariant functions between them; objects of $\mathbf{Set}^{\ll}$ will be called *absolutely continuous sets*.

There are absolutely continuous sets analogous to the sheaves RV_A of random variables and $\mathbb{P}$ of standardizable probability spaces:

Definition 4.22. For A a Polish space, a random variable $X : \mathbb{I}^\omega \to A$ has *finite footprint* if it factors through $\mathrm{proj}_{1..n}$ for some n. Let $\overline{\mathrm{RV}}_A$ be the set of random variables with finite footprint. This forms an absolutely continuous set, with action $X \cdot \pi = X \circ \pi$.

Definition 4.23. Let $\overline{\mathbb{P}}$ be the set of standardizable probability spaces on $\mathbb{I}^\omega$ with finite footprint. This forms an absolutely continuous set, with action $(\mathcal{F}, \mu) \cdot \pi = \pi^*(\mathcal{F}, \mu)$.

These yield a resource monoid in $\mathbf{Set}^{\ll}$.

Theorem 4.24. $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ is a $\mathbf{Set}^{\ll}$ resource monoid, where

- $\overline{\sqsubseteq}$: $\overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \overline{\mathrm{Prop}}$ is the map that sends $((\mathcal{G}, \mu), (\mathcal{H}, \nu))$ to $\top$ iff $\mathcal{G} \subseteq \mathcal{H}$ and $\nu|_\mathcal{G} = \mu$, where $\overline{\mathrm{Prop}}$ is the two-element set with trivial action.
- $\overline{\mathbb{P}}^2_\perp$ is the set of pairs $((\mathcal{G}, \mu), (\mathcal{H}, \nu)) \in \overline{\mathbb{P}} \times \overline{\mathbb{P}}$ for which $(\mathcal{G}, \mu)$ and $(\mathcal{H}, \nu)$ are independently combinable.
- $\bar{i}$ is the inclusion $\overline{\mathbb{P}}^2_\perp \hookrightarrow \overline{\mathbb{P}} \times \overline{\mathbb{P}}$.
- $\bar{\bullet}$: $\overline{\mathbb{P}}^2_\perp \to \overline{\mathbb{P}}$ is the map that sends independently-combinable pairs to their independent combination.
- $\overline{\mathrm{emp}} : 1 \to \overline{\mathbb{P}}$ is the constant map at the probability space f^*1 on $\mathbb{I}^\omega$ arising from enhanced pullback along the unique $\mathbf{EMS}_{\mathrm{std}}$-map $\mathbb{I}^\omega \to \mathrm{U}(1)$ into the one-point probability space 1.

4.3 The equivalence

By choosing $\mathbb{I}^\omega$ as underlying sample space and topologizing Aut $\mathbb{I}^\omega$ to permit only objects that use finitely-many dimensions of $\mathbb{I}^\omega$ at a time, we obtain continuous analogs of Homogeneity and Correspondence. This relies crucially on both the finiteness of footprints and the inclusion of negligible ideals in the base category $\mathbf{EMS}_{\mathrm{std}}$. Negligible ideals allow passing to *measure algebra* [19, 321A]:

Definition 4.25. A *measure algebra* is a tuple $(\mathfrak{A}, \overline{\mu})$ consisting of a complete Boolean algebra $\mathfrak{A}$ and a function $\overline{\mu} : \mathfrak{A} \to [0, 1]$ such that (1) $\overline{\mu}(A) > 0$ for $A \neq \perp$ and (2) $\overline{\mu}$ is countably additive in the sense that $\overline{\mu}(\bigvee_i A_i) = \sum_i \overline{\mu}(A_i)$ for all countable families $\{A_i\}_{i\in I}$ with $A_i \wedge A_j = \perp$ for all $i \neq j$. A *measure algebra homomorphism* from $(\mathfrak{A}, \overline{\mu})$ to $(\mathfrak{B}, \overline{\nu})$ is a complete Boolean algebra homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, measure-preserving in the sense that $\overline{\nu}(f(A)) = \overline{\mu}(A)$ for all $A \in \mathfrak{A}$.

Every $(X, \mathcal{F}, \mu)$ in $\mathbf{Prob}_{\mathrm{std}}$ yields a measure algebra $(\mathcal{F}/\mu, \overline{\mu})$, where $\mathcal{F}/\mu$ is the complete Boolean algebra of events $F \in \mathcal{F}$ mod $F \sim F'$ iff $\mu(F \triangle F') = 0$, and $\overline{\mu}([F]) = \mu(F)$ [19, 321H]. Every measure-preserving map f from $(X, \mathcal{F}, \mu)$ to $(Y, \mathcal{G}, \nu)$ defines a homomorphism f^* from $(\mathcal{G}/\nu, \overline{\nu})$ to $(\mathcal{F}/\mu, \overline{\mu})$ sending $[G] \in \mathcal{G}/\nu$ to $[f^{-1}(G)] \in \mathcal{F}/\mu$ [19, 324M]. This gives a duality:

Definition 4.26. A *standard probability algebra* is a measure algebra $(\mathfrak{A}, \overline{\mu})$ arising from a standard probability space as described above. Let $\mathbf{ProbAlg}_{\mathrm{std}}$ be the category of standard probability algebras and measure algebra homomorphisms between them.

Lemma 4.27. $\mathbf{Prob}_{\mathrm{std}} \simeq \mathbf{ProbAlg}^{\mathrm{op}}_{\mathrm{std}}$.

A similar duality holds also for $\mathbf{EMS}_{\mathrm{std}}$:

Definition 4.28. A *standard measurable algebra* is a complete Boolean algebra $\mathfrak{A}$ arising from a standard probability space; i.e. $\mathfrak{A}$ is isomorphic to a Boolean algebra $\mathcal{F}/\mu$ for some $(X, \mathcal{F}, \mu) \in \mathbf{Prob}_{\mathrm{std}}$. Let $\mathbf{MbleAlg}_{\mathrm{std}}$ be the category of standard measurable algebras and *injective* complete boolean algebra homomorphisms.

Lemma 4.29. $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbf{MbleAlg}^{\mathrm{op}}_{\mathrm{std}}$.

Lemmas 4.27 and 4.29 allow importing the extensive technical development of measure algebras from Fremlin [19]. In particular, the algebraic perspective reveals that the finite-footprint property from Section 4.2 is a means of producing *relatively-atomless* subalgebras:

Definition 4.30 (Fremlin [19, 331A])**.** Let $\mathfrak{A}$ be a complete Boolean algebra and $\mathfrak{B} \subseteq \mathfrak{A}$ a subalgebra. An element $a \in \mathfrak{A}$ is a $\mathfrak{B}$-*relative atom* of $\mathfrak{A}$ if the principal ideal generated by a in $\mathfrak{A}$ is $\{a \cap b \mid b \in \mathfrak{B}\}$. The algebra $\mathfrak{A}$ is $\mathfrak{B}$-*relatively atomless* if it has no $\mathfrak{B}$-relative atoms.

Theorem 4.31. Let $\mathfrak{A}$ be the measurable algebra of $\mathbb{I}^\omega$. For any $(\mathcal{G}, \mu)$ with finite footprint, $\mathfrak{A}$ is $\mathcal{G}/\mu$-relatively atomless.

Relative-atomlessness is key to obtaining continuous analogs of Homogeneity and Correspondence, which hold specifically for the case where subalgebras are relatively atomless:

Lemma 4.32 (Homogeneity)**.** For $\mathfrak{A}$ a standard measurable algebra and subalgebras $\mathfrak{B}, \mathfrak{C} \subseteq \mathfrak{A}$ that render it relatively-atomless, and a $\mathbf{MbleAlg}_{\mathrm{std}}$-morphism $f : \mathfrak{B} \hookrightarrow \mathfrak{C}$, there exists a complete Boolean algebra automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ with $\pi(b) = f(b)$ for all $b \in \mathfrak{B}$.

Lemma 4.33 (Correspondence)**.** Let $\mathfrak{A}$ be a standard measurable algebra. For any subalgebra $\mathfrak{C} \subseteq \mathfrak{A}$, let Fix $\mathfrak{C}$ be the group of $\mathfrak{A}$-automorphisms fixing every c in $\mathfrak{C}$. If $\mathfrak{A}$ is $\mathfrak{C}$-relatively atomless then Fix $\mathfrak{C} \subseteq$ Fix $\mathfrak{D}$ iff $\mathfrak{D} \subseteq \mathfrak{C}$.

These yield a continuous analog of Theorem 3.18:

Theorem 4.34. $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$.

Finally, a careful calculation across this equivalence shows that the resource monoids in Theorems 4.12 and 4.24 indeed correspond, yielding an analog of Fact 1.1 for continuous probability:

Theorem 4.35. Across $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$, the sheaf $\mathbb{P}$ corresponds to $\overline{\mathbb{P}}$, the sheaf RV_A corresponds to $\overline{\mathrm{RV}}_A$, and the resource monoid $(\sqsubseteq, \mathbb{P}\otimes\mathbb{P}, i, \bullet, \mathrm{emp})$ in $\mathbf{EMS}$ corresponds to $(\overline{\sqsubseteq}, \overline{\mathbb{P}}^2_\perp, \bar{i}, \bar{\bullet}, \overline{\mathrm{emp}})$ in $\mathbf{Set}^{\ll}$.

5 DISCUSSION & RELATED WORK

Atomic sheaves for probability. Tao [52] defines probabilistic notions as those invariant under extension of the sample space. Along these lines, Simpson [47] constructs a topos of atomic sheaves on a category of probability spaces and measure-preserving maps; in it, he presents a sheaf of random variables and an extension of the Giry monad [22] to sheaves, and shows how concepts such as independence and expectation can be internalized [46, 48].

Simpson's topos is similar to our $\mathbf{EMS}$, but our base category $\mathbf{EMS}_{\mathrm{std}}$ omits measures and its maps are quotiented by almost-everywhere equality; we instead model measures explicitly via the sheaf $\mathbb{P}$. As we have focused on separation logic, we have not investigated whether the Giry monad extends to $\mathbf{EMS}$ and the probabilistic concepts that can be expressed internally; this would make interesting future work. Simpson [47] mentions a resemblance to nominal sets, but does not extensively develop the notion to the best of our knowledge.

Simpson's topos also serves as a model of Atomic Sheaf Logic [49], a recently-developed logic axiomatizing the interaction between conditional independence and a notion of *atomic equivalence*, which in the probabilistic setting denotes equidistribution of random variables, with potential applications to developing proof-relevant probabilistic separation logics; it would be interesting to explore whether our topos admits analogous constructions.

Categorical probability. There are numerous categorical formulations of probability. Fritz [20] develops probability theory purely synthetically by axiomatizing equational properties known to hold for Markov kernels. Jackson [26], building on Breitsprecher [10], gives an alternative sheaf-theoretic model of probability by taking sheaves on a single measurable space rather than a category of measurable spaces; we speculate that there could be a relationship between this model and ours similar to the relationship between petit and gros topoi of sheaves on topological spaces [30].

Quasi-Borel spaces. The category QBS of quasi-Borel spaces [24] is a richly developed model of higher-order probability. Whereas QBS has been used extensively to model higher-order probabilistic languages [1, 44, 45, 53], our goal in constructing **EMS** and $\mathbf{Set}^{\ll}$ has been focused on refining models of probabilistic separation logic. Structurally, QBS and **EMS** are quite different: QBS is a well-pointed quasi-topos while **EMS** is a non-well-pointed topos. However, as remarked in Heunen et al. [24, Prop. 34], QBS is related to particular presheaves on the category of standard measurable spaces. This suggests connections to **EMS**, since it is a category of sheaves on $\mathbf{EMS}_{\mathrm{std}}$, but there is a gap between these two settings: $\mathbf{EMS}_{\mathrm{std}}$-morphisms are quotiented by almost-everywhere equality whereas maps of standard measurable spaces are not. We leave elucidating the relationship between our setting and QBS to future work.

General representation theorems. The equivalence $\mathbf{Sch} \simeq \mathbf{Nom}$ can be obtained via a *Fraïssé limit* [25, §7.1], a recipe for making universal objects (e.g., $\mathbb{N}$) capable of representing a class of models (e.g., finite sets). More generally, there is a long line of results giving groupoid-based representations of categories [8, 9, 11, 16, 27, 31], with a history going back to Grothendieck [17, 23]. Caramello [12] is particularly relevant, as it gives conditions closely resembling Lemmas 3.16 and 3.17 under which categories of atomic sheaves are equivalent to categories of continuous $\mathrm{Aut}(u)$-sets for suitable objects u. We are currently investigating whether Theorems 3.18 and 4.34 can be obtained via these general results, with an eye towards generalizing beyond probability to the quantum setting.

Probabilistic separation logic. PSL [6] is the first separation logic whose separating conjunction models independence, by splitting random substitutions; it has since been extended to support conditional independence [3] and negative dependence [4], and to the quantum setting [55]. In contrast to PSL and its extensions, Lilac [29] has an alternative model of separation, via independent combination. Lilac's model is complicated: independent combination is an intricate measure-theoretic operation, an intricate proof is required to show it forms a monoid, and many side conditions on this monoid are needed for soundness of Lilac's proof rules.

Theorem 4.35 simplifies and clarifies Lilac's model. It shows that independent combination arises naturally from the well-known tensor product of standard probability spaces; that independent combination forms a monoid then follows from the fact that tensor product is monoidal. The resource monoid in Theorem 4.35 replaces the side conditions on Lilac's monoid with the single notion of standardizability — a condition well-motivated by the intuition that probability spaces should arise via pullback along $\mathbf{EMS}_{\mathrm{std}}$-maps.

Theorem 4.35 also improves on the model in Li et al. [29] in multiple ways. Quotienting by negligiblity yields a model invariant under almost-everywhere equality, whereas the model in Li et al. [29] must manually track σ-ideals of negligible sets. Interpreting propositions as equivariant maps implies our model is invariant under finite-width permutations of $\mathbb{I}^{\omega}$. Finally, using the internal language of **EMS**, one can interpret quantification over propositions, allowing to generalize Lilac to a higher-order logic; in the future, we would like to explore whether this higher-order generalization can be used to specify properties of higher-order programs.

An aspect of Lilac not captured by our model is its *conditioning modality*, interpreted by disintegration [13]. This is difficult to capture in our model because $\mathbf{EMS}_{\mathrm{std}}$-objects come with a fixed collection of negligible sets, whereas disintegration can change which sets are negligible.

Probability and name generation. Recent work has identified connections between probability theory and name generation: Staton et al. [51] provides a semantics for a probabilistic language that treats random variables as dynamically-allocated read-only names, and Sabok et al. [43] show that QBS can be used to characterize observational equivalence of stateful imperative programs by interpreting dynamic allocation as probabilistic sampling. The resemblance between our probabilistic Theorem 4.35 and the store-based Fact 1.1 provides further evidence along these lines.

Nominal sets. Many constructions exist in **Nom** beyond its ability to capture permutation-invariance: *freshness quantification* [32] captures the informal convention of picking fresh names [5], a *name abstraction* [38, §4] type former gives a uniform treatment of binding, and *nominal restriction sets* [38, §9.1] models languages with locally generated names [33, 39]. It would be interesting to explore whether analogous constructions can be carried out in $\mathbf{Set}^{\ll}$, to obtain analogous treatments of the informal convention of picking fresh sample spaces [19, §27] and to provide models of probabilistic languages with locally generated random variables.

6 CONCLUSION

We unify two different approaches to separating probabilistic state: the usual product of probability spaces and independent combination. To do this, we show that separation-as-product lives in a category **EMS** of enhanced measurable sheaves, that separation-as-independent-combination lives in a category $\mathbf{Set}^{\ll}$ of absolutely continuous sets, and that these two notions of separation correspond across an equivalence $\mathbf{EMS} \simeq \mathbf{Set}^{\ll}$. This validates the use of independent combination in probabilistic separation logic [29], clarifies independent combination's relationship with traditional formulations of independence, and suggests improvements to existing models. Finally, as a probabilistic analog of **Nom**, the category $\mathbf{Set}^{\ll}$ creates new probabilistic interpretations of nominal concepts, which we hope will create more opportunities for using nominal techniques in probability.

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A OVERVIEW

The goal of this appendix is to build up towards Appendices D and E, which give sheaf- and permutation-based models of probabilistic separation logic, prove their equivalence, and show the correspondence between the tensor-product- and independent-combination-based models of probabilistic separation. To lead into this result, the appendix is structured as follows.

- Appendix B imports the measure theory and Appendix C the sheaf theory needed to present the probabilistic counterpart of Section 2. Most of the results are for handling continuous probability, but some (in particular, Lemma C.23 characterizing the Day convolution) are needed even in the discrete case. These sections are best not read linearly on first reading, but used as reference when reading Appendices D and E.
- Appendix D presents enhanced measurable sheaves.
- Appendix E presents absolutely continuous set and our probabilistic analogue of Fact 1.1.

On reading, we encourage you to follow concepts by clicking on hyperlinks that take you directly to definitions.

B GENERAL MEASURE THEORY

Throughout this section we rely heavily on Fremlin [19].

B.1 Standard probability spaces and probability algebras

Definition B.1 (standard Borel space)**.** A *standard Borel space* is a measurable space $(X, \mathcal{F})$ such that X can be made into a complete separable metrizable space (in other words, a Polish space) whose Borel σ-algebra is $\mathcal{F}$ [19, 424A].

Definition B.2 (standard probability space)**.** A *standard probability space* is a probability space $(X, \mathcal{F}, \mu)$ where $(X, \mathcal{F})$ is the completion of a standard Borel space with respect to the negligibles of μ. (For more on standard probability spaces, see Rohlin [42], where they are called Lebesgue spaces; in particular, Rohlin [42, §2.7] justifies the definition given here.)

Definition B.3 (the category $\mathbf{Prob}_{\mathrm{std}}$)**.** Let $\mathbf{Prob}_{\mathrm{std}}$ be the category whose objects are standard probability spaces and whose morphisms are measure-preserving maps quotiented by almost-sure equality:

$$\mathbf{Prob}_{\mathrm{std}}((X,\mathcal{F},\mu),(Y,\mathcal{G},\nu)) = \{f : (X,\mathcal{F}) \to (Y,\mathcal{G}) \text{ measurable} \mid \mu(f^{-1}G) = \nu(G) \text{ for all } G \text{ in } \mathcal{G}\}/{=_{\text{a.s.}}}$$
$$\text{where } f =_{\text{a.s.}} g \text{ iff } \{x \in X \mid f(x) \neq g(x)\} \text{ is } \mu\text{-negligible}$$

Definition B.4 (measure algebra)**.** A *measure algebra* is a pair $(\mathfrak{A}, \overline{\mu})$ where $\mathfrak{A}$ is a complete boolean algebra (i.e., a boolean algebra with all small meets and joins) and $\overline{\mu}$ is a function $\mathfrak{A} \to [0,1]$ satisfying

- $\overline{\mu}(\bot) = 0$
- $\overline{\mu}(a) > 0$ for all $a \neq \bot$
- $\overline{\mu}\big(\bigvee_i a_i\big) = \sum_i \overline{\mu}(a_i)$ for all sequences $(a_i)_{i\in\mathbb{N}}$ with $a_i \wedge a_j = \bot$ for all $i \neq j$.

A *probability algebra* is a measure algebra $(\mathfrak{A}, \overline{\mu})$ with $\overline{\mu}(\top) = 1$.

See Fremlin [19, 321A] for more information on measure algebras. Our definitions deviate slightly from the definitions given there: our measure-algebras are closed under all small meets and joins, so correspond to the notion of Dedekind-complete measure algebra [19, 314A(a)], whereas the definition in Fremlin [19, 321A] only requires the measure algebras to be Dedekind σ-complete (closed under countable meets and joins). This change in terminology is motivated by the fact that all probability spaces will give rise to Dedekind-complete measure algebras:

Construction B.5. Every probability space $(X, \mathcal{F}, \mu)$ gives rise to a corresponding measure algebra $(\mathfrak{A}, \overline{\mu})$ as follows. Let $\mathcal{N}$ be the σ-ideal of μ-negligible sets. Set $\mathfrak{A}$ to the quotient boolean algebra $\mathcal{F}/\mathcal{N}$. Elements of $\mathfrak{A}$ are equivalence classes $[F]$ for $F \in \mathcal{F}$, modulo $[F] = [F']$ iff $F \triangle F'$ (the symmetric difference of F and F') is μ-negligible. Define $\overline{\mu}$ by $\overline{\mu}[F] = \mu(F)$. Let $\mathrm{alg}(X, \mathcal{F}, \mu)$ denote the measure algebra constructed from a probability space $(X, \mathcal{F}, \mu)$ in this way.

For more on this construction, see Fremlin [19, 321H].

Lemma B.6. For every probability space $(X, \mathcal{F}, \mu)$ it holds that $\mathrm{alg}(X, \mathcal{F}, \mu)$ is a Dedekind-complete measure algebra in the sense of Fremlin, so a measure algebra in our sense.

Proof. $\mathrm{alg}(X, \mathcal{F}, \mathcal{N})$ is a probability algebra [19, 322A(a)], so localizable [19, 322C], so Dedekind-complete [19, 322A(e)]. □

Definition B.7 (homomorphism of measure algebras)**.** Given two measure algebras $(\mathfrak{A}, \overline{\mu})$ and $(\mathfrak{B}, \overline{\nu})$, a *measure-algebra homomorphism from* $(\mathfrak{A}, \overline{\mu})$ *to* $(\mathfrak{B}, \overline{\nu})$ is a complete-boolean-algebra homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ which preserves measures: $\overline{\nu}(f(a)) = \overline{\mu}(a)$ for all a in $\mathfrak{A}$.

See Fremlin [19, 324I] for more information on measure-algebra homomorphisms; just as our definition of measure algebra corresponds to Fremlin's Dedekind-complete measure algebra, our definition of measure-algebra homomorphism corresponds to Fremlin's order-continuous measure-algebra homomorphism.

Definition B.8 (simple product of measure algebras)**.** Given two measure algebras $(\mathfrak{A}, \overline{\mu})$ and $(\mathfrak{B}, \overline{\nu})$, their *simple product* (Fremlin [19, 322L]) is the measure algebra $(\mathfrak{A} \times \mathfrak{B}, \overline{\lambda})$ where $\mathfrak{A} \times \mathfrak{B}$ is the product boolean algebra with boolean-algebra operations computed pointwise and $\overline{\lambda}$ is the measure defined by $\overline{\lambda}(a, b) = \overline{\mu}(a) + \overline{\nu}(b)$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Since joins and meets are pointwise, the boolean algebra $\mathfrak{A} \times \mathfrak{B}$ is Dedekind-complete if $\mathfrak{A}, \mathfrak{B}$ are.

The idea is that the simple product is an algebraic counterpart to the coproduct of measure spaces: $\mathrm{alg}(X + Y) \cong \mathrm{alg}\, X \times \mathrm{alg}\, Y$. It is known that every standard probability space is isomorphic to the coproduct of countably many atoms and possibly an interval equipped with the Lebesgue measure [42, p. 25, §2]. This fact motivates the following definition:

Definition B.9 (standard probability algebra)**.** Say a probability algebra $(\mathfrak{A}, \overline{\mu})$ is *standard* if it is composed of at most countably many atoms and an interval; that is, there exists some $p \in (0, 1]$ and a countable family of weights $(q_i)_{i \in I}$ such that $(\mathfrak{A}, \overline{\mu})$ decomposes as a simple product

$$(\mathfrak{A}, \overline{\mu}) \cong ([0, p], \overline{\lambda}) \times \prod_{i \in I} \mathrm{atom}(q_i) \qquad \text{or} \qquad (\mathfrak{A}, \overline{\mu}) \cong \prod_{i \in I} \mathrm{atom}(q_i)$$

where $([0, p], \overline{\lambda})$ is the measure algebra of Lebesgue measure on the interval $[0, p]$ and $\mathrm{atom}(q_i)$ is the algebra with two elements $\bot, \top$ where $\top$ has measure q_i.

Definition B.10 (the category $\mathbf{ProbAlg}_{\mathrm{std}}$)**.** Let $\mathbf{ProbAlg}_{\mathrm{std}}$ be the category whose objects are standard probability algebras and whose morphisms are measure-algebra homomorphisms.

The operation alg extends to a functor $\mathbf{Prob}_{\mathrm{std}} \to \mathbf{ProbAlg}_{\mathrm{std}}^{\mathrm{op}}$:

Construction B.11. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be probability spaces and $\mathrm{alg}(X, \mathcal{F}, \mu) = (\mathfrak{A}, \overline{\mu})$ and $\mathrm{alg}(Y, \mathcal{G}, \nu) = (\mathfrak{B}, \overline{\nu})$ the corresponding measure algebras. Let f be a measure-preserving map $(X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$. The map $\mathrm{alg}(f) : (\mathfrak{B}, \overline{\nu}) \to (\mathfrak{A}, \overline{\mu})$ defined by setting $\mathrm{alg}(f)[G] = [f^{-1}(G)]$ for all $G \in \mathcal{G}$ is a measure-algebra homomorphism. This construction respects almost-sure equality of measure-preserving maps, sends the identity map to the identity homomorphism, and sends composition of measure-preserving maps $f \circ g$ to flipped-composition of homomorphisms $\mathrm{alg}(g) \circ \mathrm{alg}(f)$.

Proof. If $G \triangle G'$ is ν-negligible then $f^{-1}(G \triangle G') = f^{-1}(G) \triangle f^{-1}(G)$ is μ-negligible since f measure-preserving, so $\mathrm{alg}(f)$ is well-defined. Taking f-preimages distributes over all unions and intersections, so $\mathrm{alg}(f)$ is a homomorphism of complete boolean-algebras; this homomorphism preserves the measures $\overline{\mu}$ and $\overline{\nu}$ because f is measure-preserving. If $f =_{\mathrm{a.s.}} g$, then $f^{-1}(G) \triangle g^{-1}(G)$ is μ-negligible for all $G \in \mathcal{G}$, so $\mathrm{alg}(f) = \mathrm{alg}(g)$. Finally, f respects identities and composition because taking preimages does. □

Lemma B.12. The functor $\mathrm{alg} : \mathbf{Prob}_{\mathrm{std}} \to \mathbf{ProbAlg}_{\mathrm{std}}^{\mathrm{op}}$ witnesses an equivalence $\mathbf{Prob}_{\mathrm{std}} \simeq \mathbf{ProbAlg}_{\mathrm{std}}^{\mathrm{op}}$.

Proof. The functor alg is essentially surjective on objects because any standard probability algebra is isomorphic to the measure algebra generated by the coproduct of countably many atoms and possibly an interval equipped with the Lebesgue measure. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be standard probability spaces with associated probability algebras $\mathrm{alg}(X, \mathcal{F}, \mu) = (\mathfrak{A}, \overline{\mu})$ and $\mathrm{alg}(Y, \mathcal{G}, \nu) = (\mathfrak{B}, \overline{\nu})$. Fix an arbitrary measure-algebra homomorphism $f^* : (\mathfrak{B}, \overline{\nu}) \to (\mathfrak{A}, \overline{\mu})$. The probability space $(X, \mathcal{F}, \mu)$ is complete by definition and strictly localizable by Fremlin [19, 221L]. The probability space $(Y, \mathcal{G}, \nu)$ is finite, hence semi-finite [19, 221F], and nonempty. Thus the preconditions of Fremlin [19, 343B] are satisfied. As the completion of a probability measure on a standard Borel space, $(Y, \mathcal{G}, \nu)$ is a compact measure space [19, 433X(e)(i)] [19, 342G(b)], hence also locally compact [19, 342H(a)], so point (i) of Fremlin [19, 343B] holds. The implication (i)⇒(vi) of Fremlin [19, 343B] then gives a measure-preserving map $f : (X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$ such that $\mathrm{alg}(f) = f^*$. This establishes fullness of the functor alg. The space $(Y, \mathcal{G}, \nu)$ is countably separated (because it arises from a metrizable space), so by Fremlin [19, 343G] the map f is unique up to almost-sure equivalence. This establishess faithfulness of alg. □

Lemma B.13. Every map in $\mathbf{ProbAlg}_{\mathrm{std}}$ is mono.

Proof. All measure-preserving measure algebra homomorphisms are injective [19, 324K(a)]. □

Lemma B.14. Every map in $\mathbf{Prob}_{\mathrm{std}}$ is epi.

Proof. Combine Lemma B.12 and Lemma B.13. □

The following lemma seems closely connected to the theorems of Edalat [18].

Lemma B.15. The category $\mathbf{Prob}_{\mathrm{std}}$ has the right Ore property: for f, g with types as shown below, there exists a space W and maps h, k such that $fh = gk$:

$$\begin{tikzcd} (W, \mathcal{K}, \tau) \arrow[r, "k"] \arrow[d, "h"'] & (Y, \mathcal{G}, \nu) \arrow[d, "g"] \\ (X, \mathcal{F}, \mu) \arrow[r, "f"'] & (Z, \mathcal{H}, \rho) \end{tikzcd}$$

Proof. As standard probability spaces, X, Y, Z are all completions of Borel measures on Polish spaces. Let $\mathcal{F}', \mathcal{G}', \mathcal{H}'$ be the Borel sets given by the topologies on X, Y, Z respectively. Let W be the product space $X \times Y$ with Borel sets $\mathcal{K}'$ generated by rectangles $F' \times G'$ for $F' \in \mathcal{F}', G' \in \mathcal{G}'$ as usual. This product is Polish because each of its factors are. Since f, g are maps of standard probability spaces, there exist disintegrations $\{\mu|_z\}_{z \in Z}$ and $\{\nu|_z\}_{z \in Z}$. Let τ' be the function $\mathcal{K}' \to [0, 1]$ defined by

$$\tau'(K') = \int (\mu|_z \otimes \nu|_z)(K') \, \rho(\mathrm{d}z),$$

a probability measure because each $\mu|_z \otimes \nu|_z$ is. Let $(W, \mathcal{K}, \tau)$ be the completion of $(W, \mathcal{K}', \tau')$, standard because τ' is a measure on a Polish space by construction. Let h and k be the projections π_1 and π_2 respectively. These maps are measure-preserving: for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$,

$$\tau(h^{-1}(F)) = \rho(F \times Y) = \int (\mu|_z \otimes \nu|_z)(F \times Y) \, \rho(\mathrm{d}z) = \int \mu|_z(F)\nu|_z(Y) \, \rho(\mathrm{d}z) = \int \mu|_z(F) \, \rho(\mathrm{d}z) = \mu(F)$$

$$\tau(k^{-1}(G)) = \rho(X \times G) = \int (\mu|_z \otimes \nu|_z)(X \times G) \, \rho(\mathrm{d}z) = \int \mu|_z(X) \times \nu|_z(G) \, \rho(\mathrm{d}z) = \int \nu|_z(G) \, \rho(\mathrm{d}z) = \nu(G)$$

Finally, the square commutes almost-surely:

$$\Pr_{(x,y)\sim\tau}[f(h(x,y)) = g(k(x,y))] = \int (\mu|_z \otimes \nu|_z)(\{(x,y) \mid fx = gy\}) \, \rho(\mathrm{d}z)$$

$$= \int (\mu|_z \otimes \nu|_z) \left(\biguplus_{z'} (f^{-1}(z') \times g^{-1}(z')) \right) \rho(\mathrm{d}z)$$

$$= \int_{z \in \lambda} (\mu|_z \otimes \nu|_z) \left((f^{-1}(z) \times g^{-1}(z)) \cap \biguplus_{z'} (f^{-1}(z') \times g^{-1}(z')) \right) \rho(\mathrm{d}z) \text{ because } (\mu|_z \otimes \nu|_z)(f^{-1}(z) \times g^{-1}(z)) = 1 \text{ for a.a. } z$$

$$= \int (\mu|_z \otimes \nu|_z)(f^{-1}(z) \times g^{-1}(z)) \, \rho(\mathrm{d}z) = 1$$

□

B.2 Standard enhanced measurable spaces and measurable algebras

Given a standard probability space $(X, \mathcal{F}, \mu)$, one can choose to forget everything about the measure μ except for which measurable subsets are negligible, leaving behind a tuple $(X, \mathcal{F}, \mathcal{N})$ where $\mathcal{N}$ is a σ-ideal of $\mathcal{F}$. Given two standard probability spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ where μ has negligibles $\mathcal{N}$ and ν has negligibles $\mathcal{M}$, and given a morphism $[f] : (X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$ of standard probability spaces, one can choose to forget everything about the fact that $[f]$ is measure-preserving except for the fact that $\nu(G) = 0$ iff $\mu(f^{-1}(G)) = 0$, leaving behind an equivalence class $[f]$ with $f^{-1}(G) \in \mathcal{N}$ iff $G \in \mathcal{M}$ for all $G \in \mathcal{G}$. (Note this is well-defined, as only the negligible sets are needed to determine whether two morphisms f, f' are almost-surely equal.) This idea is made precise as follows.

Definition B.16 (standard enhanced measurable space)**.** An *enhanced measurable space* is a tuple $(X, \mathcal{F}, \mathcal{N})$ for which there exists a measure μ with negligibles $\mathcal{N}$ such that $(X, \mathcal{F}, \mu)$ is a probability space. A *standard enhanced measurable space* is an enhanced measurable space for which there exists a measure μ making it a standard probability space.

Definition B.17 (negligible-preserving, negligible-reflecting)**.** Given two standard measurable spaces $(X, \mathcal{F}, \mathcal{N})$ and $(Y, \mathcal{G}, \mathcal{M})$, say a measurable map $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is *negligible-reflecting* if $f^{-1}(M) \in \mathcal{N}$ for all $M \in \mathcal{M}$ and *negligible-preserving* if $f^{-1}(G) \in \mathcal{N}$ implies $G \in \mathcal{M}$ for all $G \in \mathcal{G}$.

Definition B.18 (map of standard enhanced measurable spaces)**.** Given two standard enhanced measurable spaces $(X, \mathcal{F}, \mathcal{N})$ and $(Y, \mathcal{G}, \mathcal{M})$, a *map of standard enhanced measurable spaces from* $(X, \mathcal{F}, \mathcal{N})$ *to* $(Y, \mathcal{G}, \mathcal{M})$ is a measurable map $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ that is both negligible-preserving and negligible-reflecting, quotiented by almost-sure equality: $f =_{\text{a.s.}} g$ iff $\{x \in X \mid f(x) \neq g(x)\} \in \mathcal{N}$.

Definition B.19 (the category $\mathbf{EMS}_{\text{std}}$)**.** Let $\mathbf{EMS}_{\text{std}}$ be the category of standard enhanced measurable spaces and maps between them.

The following gives a more concrete picture of what kinds of (equivalence classes of) maps live in $\mathbf{EMS}_{\text{std}}$: they can be equivalently seen as those maps that induce injective complete-boolean-algebra homomorphisms, and as measure-preserving maps that have forgotten the fact that they were measure-preserving.

Lemma B.20. Let $(X, \mathcal{F}, \mathcal{N}) \xrightarrow{f} (Y, \mathcal{G}, \mathcal{M})$ be a negligible-reflecting map of enhanced measurable spaces. The following are equivalent:

(1) f is a $\mathbf{EMS}_{\text{std}}$-map from $(X, \mathcal{F}, \mathcal{N})$ to $(Y, \mathcal{G}, \mathcal{M})$
(2) f is negligible-preserving
(3) The complete-boolean-algebra homomorphism $\mathcal{F}/\mathcal{N} \xleftarrow{\mathrm{alg}(f)} \mathcal{G}/\mathcal{M}$ described in Construction B.11 is injective
(4) There exist measures μ on $\mathcal{F}$ and ν on $\mathcal{G}$ with negligibles $\mathcal{N}$ and $\mathcal{M}$ respectively such that f is a $\mathbf{Prob}_{\text{std}}$-map $(X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$

Proof.

- (1) $\Leftrightarrow$ (2). By definition.
- (2) $\Rightarrow$ (3). If f is negligible-preserving then $\mathrm{alg}(f)[E] = \bot$ implies $[E] = \bot$, so $\mathrm{alg}(f)$ has trivial kernel.
- (3) $\Rightarrow$ (4). If $\mathrm{alg}(f) : \mathcal{F}/\mathcal{N} \hookleftarrow \mathcal{G}/\mathcal{M}$ is an injective complete-boolean-algebra homomorphism, then unforgetting a standard probability measure μ on $(X, \mathcal{F}, \mathcal{N})$ gives a standard probability space $(X, \mathcal{F}, \mu)$ and corresponding measure algebra $(\mathcal{F}/\mathcal{N}, \overline{\mu})$; restricting $\overline{\mu}$ along the injective homomorphism $\mathrm{alg}(f)$ gives a measure algebra $(\mathcal{G}/\mathcal{M}, \overline{\nu})$ making $\mathrm{alg}(f)$ a measure-algebra homomorphism $(\mathcal{F}/\mathcal{N}, \overline{\mu}) \hookleftarrow (\mathcal{G}/\mathcal{M}, \overline{\nu})$. Define $\mu : \mathcal{F} \to [0,1]$ by $\mu E = \overline{\mu}[E]$. This is a measure, since $\mu\emptyset = \overline{\mu}[\emptyset] = \overline{\mu}\bot = 0$ and if $\{E_i\}_i$ is a countable disjoint family of events then $\mu(\biguplus_i E_i) = \overline{\mu}[\biguplus_i E_i] = \overline{\mu}(\biguplus_i [E_i]) = \sum_i \overline{\mu}[E_i] = \sum_i \mu E_i$, Moreover, μ has negligibles $\mathcal{M}$, since if $N \in \mathcal{M}$ then $\mu N = \overline{\mu}[N] = \overline{\mu}\bot = 0$, and conversely if $\mu E = \overline{\mu}[E] = 0$ then $[E] = \bot$ so $E \in \mathcal{M}$. To show $(Y, \mathcal{G}, \nu)$ standard, unforget a collection of Borel sets $\mathcal{G}'$ arising from a Polish topology on Y and a measure λ making $(Y, \mathcal{G}, \lambda)$ into a standard probability space, with λ the completion of the Borel measure $\lambda|_{\mathcal{G}'}$. The measure μ is correspondingly the completion of the Borel measure $\mu|_{\mathcal{G}'}$ because $\lambda|_{\mathcal{G}'}$ and $\mu|_{\mathcal{G}'}$ have the same Borel-negligibles [19, 212E(d)], so $(Y, \mathcal{G}, \nu)$ standard.
- (4) $\Rightarrow$ (2). Any measure-preserving map is negligible-preserving.

□

Definition B.21. Let $\mathrm{U} : \mathbf{Prob}_{\mathrm{std}} \to \mathbf{EMS}_{\mathrm{std}}$ be the forgetful functor that sends a standard probability space $(X, \mathcal{F}, \mu)$ to the standard enhanced measurable space $(X, \mathcal{F}, \mathrm{negligibles}(\mu))$ and a $\mathbf{Prob}_{\mathrm{std}}$-map $[f] : (X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$ to itself (now considered as a negligible-preserving-and-reflecting map of standard enhanced measurable spaces).

Lemma B.22. The category $\mathbf{EMS}_{\mathrm{std}}$ is the image of U: the functor U is surjective on objects, and any morphism of standard enhanced measurable spaces arises from a measure-preserving map equipping those spaces with standard probability measures.

Proof. The object part follows by definition of standard enhanced measurable space. The morphism part is (1)$\Rightarrow$(4) of Lemma B.20. □

Just as there is an equivalence $\mathbf{Prob}_{\mathrm{std}} \simeq \mathbf{ProbAlg}_{\mathrm{std}}^{\mathrm{op}}$ (Lemma B.12), there is an equivalence between $\mathbf{EMS}_{\mathrm{std}}$ and a category of measure algebras that have forgotten their measures.

Definition B.23 (measurable algebra)**.** A *measurable algebra* is a complete boolean algebra $\mathfrak{A}$ for which there exists a measure $\overline{\mu} : \mathfrak{A} \to [0,1]$ making $(\mathfrak{A}, \overline{\mu})$ a probability algebra.

As with the definition of measure algebra, we deviate slightly from Fremlin [19, 391B(a)] in requiring measurable algebras to be complete as boolean algebras.

Definition B.24 (standard measurable algebra)**.** Call a measurable algebra *standard* if there exists a measure μ on it that makes it into a standard probability algebra.

Definition B.25 (the category $\mathbf{MbleAlg}_{\mathrm{std}}$)**.** Let $\mathbf{MbleAlg}_{\mathrm{std}}$ be the category of standard measurable algebras and injective complete-boolean-algebra homomorphisms.

Lemma B.26. The operation alg defines a functor $\mathbf{EMS}_{\mathrm{std}} \to \mathbf{MbleAlg}_{\mathrm{std}}^{\mathrm{op}}$ witnessing an equivalence $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbf{MbleAlg}_{\mathrm{std}}^{\mathrm{op}}$.

Proof. Forgetting about the parts of Constructions B.5 and B.11 to do with measures makes alg a functor of the required type. The proof that this functor is an equivalence is analogous to the proof of Lemma B.12. □

Corollary B.27. Every morphism in $\mathbf{EMS}_{\mathrm{std}}$ is epi.

Proof. Combine (1)$\Rightarrow$(3) from Lemma B.20 and Lemma B.26. □

Construction B.28. Suppose $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ and $(Y, \mathcal{G}, \nu) \in \mathbf{Prob}_{\mathrm{std}}$. Suppose $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \nu)$, or in other words that f is a measurable map $(X, \mathcal{F}) \to (Y, \mathcal{G})$ such that $f^{-1}(G) \in \mathcal{N}$ iff $\nu(G) = 0$. The function $f^{-1}\nu : f^{-1}\mathcal{G} \to [0,1]$ defined on the pullback σ-algebra $f^{-1}\mathcal{G} := \{f^{-1}(G) \mid G \in \mathcal{G}\}$ by $f^{-1}\nu(f^{-1}(G)) = \nu(G)$ is a probability measure, and f is measure-preserving as a map $(X, f^{-1}\mathcal{G}, f^{-1}\nu) \to (Y, \mathcal{G}, \nu)$.

Proof. The function $f^{-1}\nu$ is well-defined: if $f^{-1}(G) = f^{-1}(G')$ for some $G, G' \in \mathcal{G}$, then $f^{-1}(G \triangle G') = \emptyset \in \mathcal{N}$, so $\nu(G \triangle G') = 0$, so $\nu(G) = \nu(G')$. It indeed defines a measure, because ν is a measure and taking f-preimages preserves the empty set and complements and distributes over countable disjoint unions. Finally, f is measure-preserving by definition of $f^{-1}\nu$. □

Note B.29. The probability space $(X, f^{-1}\mathcal{G}, f^{-1}\nu)$ is not necessarily standard: for example, if f is the map $(x \mapsto [x < 1/2]) : [0,1] \to \{\top, \bot\}$ and $\{\top, \bot\}$ is given the uniform measure, then $f^{-1}\mathcal{G}$ is the atomic σ-algebra on $[0,1]$ generated by $[0, 1/2)$ and $[1/2, 1]$ and $f^{-1}\nu$ assigns each atom probability $1/2$. The triple $(X, f^{-1}\mathcal{G}, f^{-1}\nu)$ is not a standard probability space because its measure algebra has two atoms but there is no bijection from it onto the two-point space.

Lemma B.30. For $(X, \mathcal{F}, \mathcal{N}) \in \mathbf{EMS}_{\mathrm{std}}$ and $(Y, \mathcal{G}, \nu) \in \mathbf{Prob}_{\mathrm{std}}$ and $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \nu)$, there exists a measure μ extending $f^{-1}\nu$ making $(X, \mathcal{F}, \mu)$ into a standard probability space with negligibles $\mathcal{N}$ and f into a measure-preserving map $(X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$.

Proof. Unforget a measure λ with negligibles $\mathcal{N}$ such that $(X, \mathcal{F}, \lambda)$ is a standard probability space. The measure $f^{-1}\nu : f^{-1}\mathcal{G} \to [0,1]$ is absolutely continuous with respect to the restriction $\lambda|_{f^{-1}\mathcal{G}}$ of λ to the pullback σ-algebra $f^{-1}\mathcal{G}$: if $f^{-1}\nu(f^{-1}(G)) = \nu(G) = 0$ then $f^{-1}(G) \in \mathcal{N}$, so $\mu(f^{-1}(G)) = 0$. Thus $f^{-1}\nu$ has a Radon-Nikodym derivative $g : (X, f^{-1}\mathcal{G}) \to \mathbb{R}$. Since $f^{-1}\mathcal{G} \subseteq \mathcal{F}$, the derivative g is also measurable as a function $(X, \mathcal{G}) \to \mathbb{R}$. Let μ be the measure $\mathcal{F} \to [0,1]$ defined by $\mu(F) = \int [x \in F] g(x)\, \lambda(\mathrm{d}x)$ for all $F \in \mathcal{F}$. This extends $f^{-1}\nu$, and so makes f measure-preserving as a map $(X, \mathcal{F}, \mu) \to (Y, \mathcal{G}, \nu)$.

All that's left is to show $(X, \mathcal{F}, \mu)$ standard with negligibles $\mathcal{N}$. Since g is measurable as a function $(X, f^{-1}\mathcal{G}) \to \mathbb{R}$, there must be some $G \in \mathcal{G}$ with $\{x \mid g(x) = 0\} = g^{-1}(0) = f^{-1}(G)$. This G must be ν-negligible, since

$$\nu(G) = f^{-1}\nu(f^{-1}(G)) = \int [f(x) \in G] g(x)\, \lambda_{f^{-1}\mathcal{G}}(\mathrm{d}x) = \int [g(x) = 0] g(x)\, \lambda_{f^{-1}\mathcal{G}}(\mathrm{d}x) = 0.$$

Since f is negligible-reflecting, this implies $f^{-1}(G) = \{x \mid g(x) = 0\}$ is λ-negligible, so the derivative g is λ-almost-everywhere strictly positive. This implies $\nu(F) = \int [x \in F] g(x)\, \lambda(\mathrm{d}x) = 0$ iff $\lambda(F) = 0$, so λ and μ have the same negligible sets $\mathcal{N}$. Finally, unforget a collection of Borel sets $\mathcal{F}'$ arising from a Polish topology on X such that λ is the completion of the Borel measure $\lambda|_{\mathcal{F}'}$. The measure μ is correspondingly the completion of the Borel measure $\mu|_{\mathcal{F}'}$ because $\lambda|_{\mathcal{F}'}$ and $\mu|_{\mathcal{F}'}$ have the same Borel-negligibles [19, 212E(d)], so $(X, \mathcal{F}, \mu)$ standard. □

The following lemma can also be proved via a mild strengthening of Wendt [54, Definition 3.1].

Lemma B.31. The category $\mathbf{EMS}_{\mathrm{std}}$ has the right Ore property: for f, g with types as shown below, there exists a space W and maps h, k such that $fh = gk$:

$$\begin{tikzcd} (W, \mathcal{K}, \mathcal{S}) \arrow[r, "k"] \arrow[d, "h"'] & (Y, \mathcal{G}, \mathcal{M}) \arrow[d, "g"] \\ (X, \mathcal{F}, \mathcal{N}) \arrow[r, "f"'] & (Z, \mathcal{H}, \mathcal{R}) \end{tikzcd}$$

Proof. By (1)⟹(4) of Lemma B.20, there exist measures μ and ρ with negligibles $\mathcal{N}$ and $\mathcal{R}$ respectively making f a $\mathbf{Prob}_{\mathrm{std}}$-map from $(X, \mathcal{F}, \mu)$ to $(Z, \mathcal{H}, \rho)$. By Lemma B.30 there exists a measure ν with negligibles $\mathcal{M}$ making $(Y, \mathcal{G}, \nu)$ a standard probability space and g a $\mathbf{Prob}_{\mathrm{std}}$-map from $(Y, \mathcal{G}, \nu)$ to $(Z, \mathcal{H}, \rho)$. The result follows by applying Lemma B.15 and forgetting all measures. □

B.2.1 *Semicartesian structure of* $\mathbf{EMS}_{\mathrm{std}}$.

Fact B.32 (semicartesian structure of $\mathbf{Prob}_{\mathrm{std}}$)**.** The category $\mathbf{Prob}_{\mathrm{std}}$ of standard probability spaces and measure-preserving maps is symmetric semicartesian monoidal. The symmetric monoidal product of two standard probability spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ has underlying set $X \times Y$ with σ-algebra generated by rectangles $F \times G$ for $F \in \mathcal{F}, G \in \mathcal{G}$, and probability measure defined by $(\mu \otimes \nu)(E) = \iint [(x, y) \in E]\, \mathrm{d}\mu(x)\mathrm{d}\nu(y)$ [2, Definition 5.25]. This preserves standardness [42, p. 37, §2.7]. The unit is the one-point probability space.

The fact that $\mathbf{EMS}_{\mathrm{std}}$ is the image of U (Lemma B.22) suggests the semicartesian symmetric monoidal structure on $\mathbf{EMS}_{\mathrm{std}}$ ought to be the image of the corresponding structure on $\mathbf{Prob}_{\mathrm{std}}$. This is indeed the case, as we now show.

Lemma B.33. Let X be an object of $\mathbf{EMS}_{\mathrm{std}}$ and X_1, X_2 two objects in $\mathbf{Prob}_{\mathrm{std}}$ that forget to it (i.e., $\mathrm{U}X_1 = \mathrm{U}X_2 = X$). Similarly let Y be an object of $\mathbf{EMS}_{\mathrm{std}}$ and Y_1, Y_2 two objects of $\mathbf{Prob}_{\mathrm{std}}$ with $\mathrm{U}Y_1 = \mathrm{U}Y_2 = Y$. Then the monoidal products $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ forget to the same standard enhanced measurable space: $\mathrm{U}(X_1 \otimes Y_1) = \mathrm{U}(X_2 \otimes Y_2)$.

Proof. Both $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ have the same underlying set (given by set-theoretic product) and σ-algebra (generated by rectangles), so all that's left is to show that the measures on $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ have the same negligible sets. Let μ_{X_1} be the measure on X_1 and μ_{X_2} the measure on X_2. Since $\mathrm{U}(X_1) = \mathrm{U}(X_2) = X$, the measures μ_{X_1} and μ_{X_2} have the same negligibles, so μ_{X_2} is absolutely continuous with respect to μ_{X_1} and has a Radon-Nikodym derivative $f : X \to \mathbb{R}$. The derivative f can be taken to be strictly positive: being a derivative forces $f >_{\mu_{X_1}\text{-a.e.}} 0$, in or other words that $E = \{x \mid f(x) = 0\}$ is μ_{X_1}-negligible, for otherwise we would have $\mu_{X_2}(E) = \int [x \in E] f(x)\, \mathrm{d}\mu_{X_1}(x) = 0$ contradicting the hypothesis that μ_{X_1} and μ_{X_2} have the same negligible sets. Running the same argument on $\mathrm{U}(Y_1) = \mathrm{U}(Y_2) = Y$ shows μ_{Y_2} has a strictly positive Radon-Nikodym derivative $g : X \to \mathbb{R}$ with respect to μ_{X_1}. Therefore the product measure $\mu_{X_2} \otimes \mu_{Y_2}$ has Radon-Nikodym derivative $h(x, y) = f(x)g(y)$ with respect to product measure $\mu_{X_1} \otimes \mu_{Y_1}$, strictly positive because f, g are, and by Fubini's theorem

$$(\mu_{X_2} \otimes \mu_{Y_2})(E) = \iint [(x, y) \in E]\, \mathrm{d}\mu_{X_2}(x)\mathrm{d}\mu_{Y_2}(y) = \iint [(x, y) \in E] f(x)g(y)\, \mathrm{d}\mu_{X_1}(x)\mathrm{d}\mu_{Y_1}(y)$$

is zero exactly when $(\mu_{X_1} \otimes \mu_{Y_1})(E)$ is, as required. □

Definition B.34 (tensor product of standard enhanced measurable spaces)**.** The *tensor product* of standard enhanced measurable spaces X, Y, written $X \otimes Y$, is defined to be $\mathrm{U}(X' \otimes_{\mathbf{Prob}_{\mathrm{std}}} Y')$. where X' and Y' are arbitrary standard probability spaces with $\mathrm{U}(X') = X$ and $\mathrm{U}(Y') = Y$. This is well-defined by Lemma B.33: the choice of X', Y' does not matter.

This operation extends to a functor $-_1 \otimes -_2 : \mathbf{EMS}_{\mathrm{std}} \times \mathbf{EMS}_{\mathrm{std}} \to \mathbf{EMS}_{\mathrm{std}}$: given $\mathbf{EMS}_{\mathrm{std}}$-maps $f : X \to Y$ and $g : A \to B$, pick arbitrary $X', Y' \in \mathbf{Prob}_{\mathrm{std}}$ with $\mathrm{U}(X') = X$ and $\mathrm{U}(Y') = Y$ making f into a $\mathbf{Prob}_{\mathrm{std}}$-morphism (which exist by Lemma B.22) and similarly pick A', B'

for g, then set $f \otimes g$ to be the map $\mathrm{U}(f \otimes_{\mathbf{Prob}_{\mathrm{std}}} g) : X \otimes A \to Y \otimes B$. The choice of X', Y', A', B' doesn't matter because U faithful. Functoriality follows from functoriality of $\otimes_{\mathbf{Prob}_{\mathrm{std}}}$.

Lemma B.35. The category $\mathrm{EMS}_{\mathrm{std}}$ is semicartesian monoidal with symmetric monoidal product $(\otimes)$.

Proof. The associator, unitor, and swapping map are the images under U of their counterparts in $\mathbf{Prob}_{\mathrm{std}}$ (Fact B.32). The coherence diagrams commute because they are the image under U of the corresponding diagrams for $\otimes_{\mathbf{Prob}_{\mathrm{std}}}$. □

B.3 Subalgebras with enough room

Throughout this section, when we say "subalgebra" we mean what Fremlin [19] calls "closed subalgebra". There are two possible meanings for the word "closed", one order-theoretic and one topological, but thankfully the two coincide in our situation: Fremlin [19, 323H] shows "closed" has a canonical meaning for localizable (hence also for probability) algebras. We will use the order-theoretic definition, so a closed subalgebra is a boolean subalgebra closed under all small joins [19, 313D(a)].

Definition B.36 (enough room)**.** Say a subalgebra $\mathfrak{C}$ of $[0, 1]$ has *enough room* if every standard probability algebra is isomorphic to a subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ such that $\mathfrak{C}$ and $\mathfrak{D}$ are stochastically independent [19, 325L].

Intuition B.37. A subalgebra $\mathfrak{C}$ of $[0, 1]$ having enough room is analogous to Lilac's requirement that sub-σ-algebras of the Hilbert cube have finite footprint [29, Definition 2.6]. The main idea is that, if $[0, 1]$ is to be an inexhaustible source of randomness, then a given subalgebra must be small enough that one can always allocate fresh (i.e., stochastically independent) standard probability algebras inside of it.

Note B.38. We have a lot of freedom in how we choose to picture the measure algebra $[0, 1]$. As a measure algebra, it is isomorphic to $[0, 1]^2$ (the Lebesgue measure on the unit square) and $[0, 1/2] \times [0, 1/2]$ and $[0, 1/3] \times [0, 2/3]$ (where $\times$ is the simple product of measure algebras, so that these algebras correspond to the disjoint union of two intervals) and so on. Because of this, in pictures below $[0, 1]$ may be drawn as any of these spaces.

B.3.1 A subalgebra with enough room.

Example B.39. Let $\mathfrak{A}$ be the probability algebra given by the Lebesgue measure on the unit square $[0, 1]^2$. Let $\mathfrak{C}$ be the subalgebra generated by the projection $\pi_1 : [0, 1]^2 \to [0, 1]$, corresponding to the sub-σ-algebra consisting of rectangles of the form $F \times [0, 1]$ for F a Lebesgue-measurable subset of $[0, 1]$. The subalgebra $\mathfrak{C}$ has enough room, because any standard probability algebra can be embedded as a subalgebra of $\mathfrak{A}$ corresponding to a sub-σ-algebra consisting of rectangles of the form $[0, 1] \times F'$.

B.3.2 Characterizing subalgebras with enough room. We now formalize the intuitions sketched in the above example, and characterize the subalgebras with enough room.

Lemma B.40. Let $\mathfrak{C}$ be a subalgebra of a measure algebra $(\mathfrak{A}, \mu)$ and a an element of $\mathfrak{A}$ with nontrivial measure (so $\mu(a) \notin \{0, 1\}$) that is independent of $\mathfrak{C}$ in the sense that $\mu(a \cap c) = \mu(a)\mu(c)$ for all c in $\mathfrak{C}$. Then a is incomparable with every element of $\mathfrak{C}$; that is, for all c nontrivial in $\mathfrak{C}$ we have neither $a \subseteq c$ nor $a \supseteq c$.

Proof. If $a \subseteq c$ for some nontrivial c then $\mu(a) = \mu(a \cap c) = \mu(a)\mu(c)$ forcing $\mu(c) = 0$ or $\mu(a) = 1$, impossible because a, c nontrivial. Similarly, if $a \supseteq c$ for some nontrivial c then $\mu(c) = \mu(a \cap c) = \mu(a)\mu(c)$ forcing $\mu(a) = 0$ or $\mu(c) = 1$, impossible because a, c nontrivial. □

Lemma B.41. A subalgebra $\mathfrak{C}$ of $[0, 1]$ has enough room iff there exists a subalgebra $\mathfrak{D}$ isomorphic to $[0, 1]$ with $\mathfrak{C}, \mathfrak{D}$ independent.

Proof. If $\mathfrak{C}$ has enough room then there certainly is an independent subalgebra isomorphic to $[0, 1]$, since $[0, 1]$ is a standard probability algebra. Conversely, any standard probability algebra $\mathfrak{F}$ can be embedded as subalgebra of $[0, 1]$, so if $\mathfrak{C}$ is independent of $\mathfrak{D}$ isomorphic to $[0, 1]$ then the subalgebra of $\mathfrak{D}$ corresponding to $\mathfrak{F}$ is also independent of $\mathfrak{C}$. □

Lemma B.42. If $\mathfrak{C}$ has enough room and $\mathfrak{B} \subseteq \mathfrak{C}$ then $\mathfrak{B}$ has enough room.

Proof. There exists $\mathfrak{D}$ with $\mathfrak{D} \cong [0, 1]$ and $\mathfrak{C}, \mathfrak{D}$ independent, so $\mathfrak{B}, \mathfrak{D}$ independent, so $\mathfrak{B}$ has enough room by Lemma B.41. □

Lemma B.43. If $\mathfrak{C}$ has enough room and π is an automorphism of $[0, 1]$ then $\pi\mathfrak{C}$ has enough room.

Proof. There exists $\mathfrak{D}$ with $\mathfrak{D} \cong [0, 1]$ and $\mathfrak{C}, \mathfrak{D}$ independent, so $\pi\mathfrak{C}, \pi\mathfrak{D}$ independent and $\pi\mathfrak{D} \cong [0, 1]$, so $\pi\mathfrak{B}$ has enough room by Lemma B.41. □

Lemma B.44. If a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ has enough room then for any standard probability algebra there exists a subalgebra $\mathfrak{D}$ isomorphic to it and independent of $\mathfrak{C}$ such that the subalgebra generated by $\mathfrak{C}, \mathfrak{D}$ still has enough room.

Proof. Since $\mathfrak{C}$ has enough room, there is a copy of $[0,1]^2$ embedded in $\mathfrak{A}$ independent of $\mathfrak{C}$. Any standard probability space can be embedded as a subalgebra $\mathfrak{D}$ of $[0,1]$, hence as a subalgebra $\mathfrak{D} \hat{\otimes} \top$ of $[0,1]^2 = [0,1] \hat{\otimes} [0,1]$ [19, 325F], where $\hat{\otimes}$ is the localizable measure algebra free product [19, 325E]. By associativity of independent subalgebras [19, 272K] [19, 325X(g)] this implies the subalgebra generated by $\mathfrak{C}, \mathfrak{D}$ is independent of the subalgebra $\top \hat{\otimes} [0,1] \cong [0,1]$ of the copy of $[0,1]^2$ embedded in $\mathfrak{C}$. Thus we have found $\mathfrak{D}$ such that the subalgebra generated by $\mathfrak{C}, \mathfrak{D}$ is independent of a copy of $[0,1]$, which implies the subalgebra generated by $\mathfrak{C}, \mathfrak{D}$ has enough room by Lemma B.41. □

Lemma B.45. If $\mathfrak{C} \subseteq [0,1]$ has enough room, then $[0,1]$ is relatively atomless over $\mathfrak{C}$ [19, 331A].

Proof. Since $\mathfrak{C}$ has enough room, there is a subalgebra $\mathfrak{D}$ of $[0,1]$ with $\mathfrak{D} \cong [0,1]$ and $\mathfrak{C}, \mathfrak{D}$ independent. Let $\langle \mathfrak{C}, \mathfrak{D} \rangle$ be the subalgebra generated by $\mathfrak{C}$ and $\mathfrak{D}$. Since $\mathfrak{D} \cong [0,1]$, there is an isomorphism $\pi : \langle \mathfrak{C}, \mathfrak{D} \rangle \cong \mathfrak{C} \hat{\otimes} \mathfrak{D} \cong \mathfrak{C} \hat{\otimes} [0,1]$ [19, 325L]. Under this isomorphism, elements c of $\mathfrak{C}$ look like rectangles $\pi c \hat{\otimes} \top_{[0,1]}$, and the image of π is the subalgebra $\{c \hat{\otimes} \top \mid c \in [0,1]\} \subseteq \mathfrak{C} \hat{\otimes} [0,1]$ of all such rectangles. The measure algebra $\mathfrak{C} \hat{\otimes} [0,1]$ is relatively atomless over this subalgebra, because the principal ideal generated by any rectangle $c \hat{\otimes} \top$ contains elements such as $c \hat{\otimes} [0,1/2]$ that are not of the form $(c \hat{\otimes} \top) \cap (d \hat{\otimes} \top)$ for any d. Thus $\langle \mathfrak{C}, \mathfrak{D} \rangle$ is relatively atomless over $\mathfrak{C}$. Since $\mathfrak{C} \subseteq \langle \mathfrak{C}, \mathfrak{D} \rangle \subseteq [0,1]$, this implies $[0,1]$ relatively atomless over $\mathfrak{C}$ [19, 331Y(a)]. □

Lemma B.46. If $[0,1]$ is $\mathfrak{C}$-relatively atomless, then there is an isomorphism $\pi : [0,1] \to \mathfrak{C} \hat{\otimes} [0,1]$ with $\pi c = c \hat{\otimes} \top_{[0,1]}$ for all c in $\mathfrak{C}$.

Proof. Consider the decomposition

$$\pi : [0,1] \xrightarrow{\sim} \prod_{n \in \mathbb{N}} \mathfrak{C}_{e_n} \times \prod_{\kappa \in K} \mathfrak{C}_{e_\kappa} \hat{\otimes} \mathfrak{B}_\kappa$$
$$\text{for all } c \text{ in } \mathfrak{C}, \quad \pi c = ((c \cap e_n)_{n \in \mathbb{N}}, ((c \cap e_\kappa) \hat{\otimes} \top_{\mathfrak{B}_\kappa})_{\kappa \in K})$$

of $[0,1]$ with respect to $\mathfrak{C}$ given by Fremlin [19, 333K], where $\mathfrak{B}_\kappa$ denotes the standard measure algebra on $\{0,1\}^\kappa$ [19, 333A(d)]. (We have changed the notation slightly from the statement of this decomposition in Fremlin, writing $\top_{\mathfrak{B}_\kappa}$ for the top element of the algebra $\mathfrak{B}_\kappa$ instead of 1, to avoid confusion with the real number 1 in $[0,1]$.) As discussed in Fremlin [19, 333K(a)], each factor $\mathfrak{C}_{e_n}$ corresponds to a relative atom of $[0,1]$ over $\mathfrak{C}$. Since $[0,1]$ is $\mathfrak{C}$-relatively atomless, we must have $e_n = \perp$ for all n, so that the product $\prod_n \mathfrak{C}_{e_n}$ disappears and π can be rewritten as

$$\pi : [0,1] \xrightarrow{\sim} \prod_{\kappa \in K} \mathfrak{C}_{e_\kappa} \hat{\otimes} \mathfrak{B}_\kappa$$
$$\text{for all } c \text{ in } \mathfrak{C}, \quad \pi c = ((c \cap e_\kappa) \hat{\otimes} 1)_{\kappa \in K}$$

Each factor $\mathfrak{C}_{e_\kappa} \hat{\otimes} \mathfrak{B}_\kappa$ corresponds to a principal ideal of $[0,1]$ with Maharam type κ [19, 333G(a)]. The measure algebra $[0,1]$ is Maharam-type-homogenous with Maharam type ω [19, 331K] [19, 254K], so does not have any principal ideals with Maharam type $\kappa > \omega$ [19, 331H(c)]. Thus, if π is to be an isomorphism, we must have $K = \{\omega\}$, so the product $\prod_\kappa \mathfrak{C}_{e_\kappa} \hat{\otimes} \mathfrak{B}_\kappa$ reduces to $\mathfrak{C}_{e_\omega} \hat{\otimes} \mathfrak{B}_\omega$. Moreover, the measure algebra $\mathfrak{B}_\omega$ is isomorphic to $[0,1]$ [19, 254K]. Thus π can be rewritten as

$$\pi : [0,1] \xrightarrow{\sim} \mathfrak{C}_{e_\omega} \hat{\otimes} [0,1]$$
$$\text{for all } c \text{ in } \mathfrak{C}, \quad \pi c = (c \cap e_\omega) \hat{\otimes} \top_{[0,1]}$$

Finally, the fact that π preserves measure forces $\mu(e_\omega) = 1$; this in turn forces e_ω to have measure 1, and so $e_\omega = \top_{\mathfrak{C}}$. Thus

$$\pi : [0,1] \xrightarrow{\sim} \mathfrak{C} \hat{\otimes} [0,1]$$
$$\text{for all } c \text{ in } \mathfrak{C}, \quad \pi c = c \hat{\otimes} \top_{[0,1]}$$

as claimed. □

Lemma B.47. If there is an isomorphism $\pi : [0,1] \to \mathfrak{C} \hat{\otimes} [0,1]$ with $\pi c = c \hat{\otimes} \top_{[0,1]}$ for all c in $\mathfrak{C}$, then $\mathfrak{C}$ has enough room.

Proof. The subalgebra $\top_{\mathfrak{C}} \hat{\otimes} [0,1]$ generated by rectangles $\top_{\mathfrak{C}} \hat{\otimes} a$ for a in $[0,1]$ is isomorphic to $[0,1]$ and independent of $\mathfrak{C} \hat{\otimes} \top_{[0,1]}$. Transporting back along π gives a subalgebra $\pi^{-1}(\top_{\mathfrak{C}} \hat{\otimes} [0,1])$ of $[0,1]$ independent of $\mathfrak{C}$, so $\mathfrak{C}$ has enough room by Lemma B.41. □

Theorem B.48 (characterization of subalgebras with enough room)**.** The following are equivalent:

(1) $\mathfrak{C} \subseteq [0,1]$ has enough room
(2) There is an isomorphism $\pi : [0,1] \to \mathfrak{C} \hat{\otimes} [0,1]$ with $\pi c = c \hat{\otimes} \top_{[0,1]}$ for all c in $\mathfrak{C}$
(3) $[0,1]$ is $\mathfrak{C}$-relatively atomless

Proof. Lemma B.45, Lemma B.46, and Lemma B.47 give the cycle $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. □

Intuition B.49. Theorem B.48 says subalgebras with enough room have a very nice form: up to measure-algebra isomorphism, they are projections out of a product space with an interval's worth of independence available for allocating fresh randomness. Also, the characterization based on relative-atomlessness shows that the idea of having enough room does not depend on measures in an essential way, as relative-atomlessness is purely a property of the underlying Boolean algebras.

B.4 Automorphisms of the interval

Lemma B.50. If $[0,1]$ is relatively atomless over a subalgebra $\mathfrak{C}$, then $\tau_{\mathfrak{C}_a}([0,1]_a) = \omega$ for all $a \neq \bot$ in $[0,1]$.

Proof. Relative atomlessness implies $\tau_{\mathfrak{C}_a}[0,1]_a \geq \omega$ for all $a \neq \bot$ in $[0,1]$ [19, 333B(d)], and

$$\tau_{\mathfrak{C}_a}[0,1]_a \overset{\text{333B(a)}}{\leq} \tau_{\mathfrak{C}}[0,1] \overset{\text{333B(e)}}{\leq} \tau[0,1] \overset{\text{331K+254K}}{=} \omega.$$

□

Lemma B.51. Let $\mathfrak{A}$ be a subalgebra of $[0,1]$ considered as a measurable algebra. If $[0,1]$ is $\mathfrak{A}$-relatively atomless, then there exists a measurable-algebra isomorphism $\pi : [0,1] \to \mathfrak{A} \mathbin{\hat{\otimes}} [0,1]$ such that $\pi a = a \mathbin{\hat{\otimes}} \top_{[0,1]}$ for all a in $\mathfrak{A}$.

Proof. Put the usual Lebesgue measure λ on $[0,1]$, rendering $\mathfrak{A}$ a closed sub-measure-algebra of the measure algebra $([0,1],\lambda)$, apply Lemma B.46 to get a measure algebra isomorphism $\pi : ([0,1],\lambda) \to (\mathfrak{A}, \lambda|_{\mathfrak{A}}) \mathbin{\hat{\otimes}} ([0,1],\lambda)$ with $\pi a = a \mathbin{\hat{\otimes}} \top_{[0,1]}$ for all a in $\mathfrak{A}$, and then forget all the measures and the fact that π preserves them. □

Lemma B.52 (Homogeneity)**.** Let $\mathfrak{A}, \mathfrak{B}$ be standard probability subalgebras of $[0,1]$ that render it relatively atomless. Let $f : \mathfrak{B} \hookrightarrow \mathfrak{A}$ be an injective order-continuous Boolean algebra homomorphism. There exists an order-continuous Boolean algebra automorphism $\pi : [0,1] \to [0,1]$ refining f in the sense that $\pi b = fb$ for all b in $\mathfrak{B}$. In other words, the diagram

$$\begin{tikzcd}
{[0,1]} & {[0,1]} \arrow[l, dashed, "\pi"'] \\
\mathfrak{A} \arrow[u, hook] & \mathfrak{B} \arrow[u, hook] \arrow[l, hook', "f"]
\end{tikzcd}$$

commutes, where the vertical arrows are the inclusions $\mathfrak{A} \subseteq [0,1]$ and $\mathfrak{B} \subseteq [0,1]$.

Proof. Give $[0,1]$ in the top-left corner the usual Lebesgue measure λ. Since f and the inclusion $\mathfrak{A} \subseteq [0,1]$ are injective, restricting λ along these gives measures μ, ν such that

$$\begin{tikzcd}
{([0,1],\lambda)} & \\
(\mathfrak{A},\mu) \arrow[u, hook] & (\mathfrak{B},\nu) \arrow[l, hook', "f"]
\end{tikzcd}$$

is a diagram in **ProbAlg**$_{\mathrm{std}}$, the category of standard probability algebras and measure-algebra homomorphisms. By Lemma B.51, there exists an order-continuous Boolean algebra isomorphism $\pi : [0,1] \to \mathfrak{B} \mathbin{\hat{\otimes}} [0,1]$ such that the triangle

$$\begin{tikzcd}
\mathfrak{B} \mathbin{\hat{\otimes}} [0,1] & {[0,1]} \arrow[l, two heads, hook', "\pi"'] \\
 & \mathfrak{B} \arrow[u, hook] \arrow[ul, "- \hat{\otimes} \top_{[0,1]}"]
\end{tikzcd}$$

commutes. The homomorphism $- \mathbin{\hat{\otimes}} \top_{[0,1]}$ is measure-preserving as a map $(\mathfrak{B},\nu) \to (\mathfrak{B},\nu) \mathbin{\hat{\otimes}} ([0,1],\lambda)$, since

$$(\nu \mathbin{\hat{\otimes}} \lambda)(b \mathbin{\hat{\otimes}} \top_{[0,1]}) = \nu b \cdot \lambda \top_{[0,1]} = \nu b$$

for all b in $\mathfrak{B}$. Therefore it fits in with the other **ProbAlg**$_{\mathrm{std}}$-diagram above to give the following **ProbAlg**$_{\mathrm{std}}$-diagram:

$$\begin{tikzcd}
{([0,1],\lambda)} & (\mathfrak{B},\nu) \mathbin{\hat{\otimes}} ([0,1],\lambda) & \\
(\mathfrak{A},\mu) \arrow[u, hook] & & (\mathfrak{B},\nu) \arrow[ll, hook', "f"] \arrow[ul, "- \hat{\otimes} \top_{[0,1]}"']
\end{tikzcd}$$

This diagram can be completed into a commutative quadrilateral. Note that $[0,1]$ is $\mathfrak{A}$-relatively atomless and $(\mathfrak{B},\nu) \mathbin{\hat{\otimes}} ([0,1],\lambda)$ is $\mathfrak{B}$-relatively atomless, and $(\mathfrak{B},\nu) \mathbin{\hat{\otimes}} ([0,1],\lambda)$ has countable Maharam type [19, 333G(a)] because $\mathfrak{B}$ standard and so has at-most-countable Maharam type. Therefore, the same argument as in Lemma B.50 gives $\tau_{[0,1]_a}([0,1]) = \tau_{(\mathfrak{B}\hat{\otimes}[0,1])_{b\hat{\otimes}\top}}(\mathfrak{B}\hat{\otimes}[0,1]) = \omega$ for all a in $[0,1]$ and b in $\mathfrak{B}$. Now Fremlin [19, 333C(b)] gives a measure algebra isomorphism σ completing the above diagram into a commutative quadrilateral:

$$\begin{tikzcd}
([0,1],\lambda) & & (\mathfrak{B},\nu)\hat{\otimes}([0,1],\lambda) \arrow[ll, two heads, hook, dashed, "\sigma"'] & \\
(\mathfrak{A},\mu) \arrow[u, hook] & & & (\mathfrak{B},\nu) \arrow[lll, hook', "f"] \arrow[ul, "-\hat{\otimes}\top_{[0,1]}"']
\end{tikzcd}$$

Forgetting all measures and combining this quadrilateral with the triangle above gives the following commutative rectangle of measurable algebras:

$$\begin{tikzcd}
{[0,1]} & \mathfrak{B}\hat{\otimes}[0,1] \arrow[l, two heads, hook, "\sigma"'] & {[0,1]} \arrow[l, two heads, hook, "\pi"'] \\
\mathfrak{A} \arrow[u, hook] & & \mathfrak{B} \arrow[ll, hook', "f"] \arrow[ul, "-\hat{\otimes}\top"'] \arrow[u, hook]
\end{tikzcd}$$

The composite $\sigma\pi$ is an automorphism of the form required. ◻

Lemma B.53 (Correspondence)**.** For $\mathfrak{C} \subseteq [0,1]$, let Fix $\mathfrak{C}$ be the subgroup of $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}[0,1]$ consisting of autos fixing every c in $\mathfrak{C}$:

$$\mathrm{Fix}\,\mathfrak{C} := \{\pi \mid \pi c = c \text{ for all } c \in \mathfrak{C}\}.$$

If $[0,1]$ is $\mathfrak{C}$-relatively atomless and Fix $\mathfrak{C} \subseteq$ Fix $\mathfrak{D}$, then $\mathfrak{D} \subseteq \mathfrak{C}$.

Proof. If every $\mathbf{EMS}_{\mathrm{std}}$-auto fixing $\mathfrak{C}$ fixes $\mathfrak{D}$, then surely every $\mathbf{Prob}_{\mathrm{std}}$-auto fixing $\mathfrak{C}$ fixes $\mathfrak{D}$. Now suppose for contradiction that there exists d in $\mathfrak{D}$ not in $\mathfrak{C}$, with aim to construct a $\mathbf{Prob}_{\mathrm{std}}$-auto fixing $\mathfrak{C}$ but not d. By Theorem B.48, there is an isomorphism $\sigma : [0,1] \to \mathfrak{C}\hat{\otimes}[0,1]$ such that $\sigma c = c \hat{\otimes} \top_{[0,1]}$ for all c in $\mathfrak{C}$. By Fremlin [19, 333R(ii)⇒(iii)], $\mathfrak{C}$ arises as the fixed-point subalgebra of a particular automorphism $\pi : [0,1] \xrightarrow{\sim} [0,1]$, so that $c \in \mathfrak{C}$ iff $\pi c = c$. Since $d \notin \mathfrak{C}$ we must have $\pi d \neq d$, so π is an auto of the form required. ◻

B.5 The target practice lemma

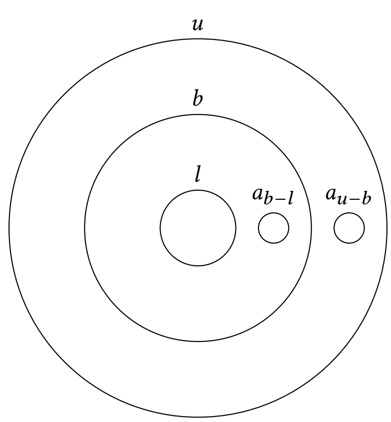
Lemma B.54. Let $\mathfrak{B}$ be a standard measurable algebra, $\mathfrak{A} \subseteq \mathfrak{B}$ a standard measurable subalgebra, and b an element of $\mathfrak{B}$ not in $\mathfrak{A}$. There exists a standard measurable algebra $\mathfrak{C}$ and homomorphisms of standard measurable algebras $f, g : \mathfrak{B} \hookrightarrow \mathfrak{C}$ such that $f|_{\mathfrak{A}} = g|_{\mathfrak{A}}$ and $f(b) \notin \operatorname{im} g$.

Proof. Let l be the largest element of $\mathfrak{A}$ contained in b and u the smallest element of $\mathfrak{A}$ containing b:

$$l := \bigvee_{\mathfrak{A}\ni a \leq b} a \qquad u := \bigwedge_{\mathfrak{A}\ni a \geq b} a$$

These exist and are elements of $\mathfrak{A}$ because $\mathfrak{A}$ is a complete boolean algebra. Since b is not in $\mathfrak{A}$, neither l nor u can be equal to it. Thus b is sandwiched between l and u by a sequence of strict inequalities $l < b < u$.

Consider an arbitrary a in $\mathfrak{A}$ less than $u - l$. Any such a can be decomposed into a disjoint union $a_{u-b} + a_{b-l}$ with $a_{u-b} \leq u - b$ and $a_{b-l} \leq b - l$, by setting $a_{u-b} := a \wedge (u - b)$ and $a_{b-l} := a \wedge (b - l)$. The following diagram depicts the situation (and explains the lemma's name):



The idea is to make use of the following fact: for any a in $\mathfrak{A}$ that is nonzero (i.e., $a \neq \bot$), both a_{u-b} and a_{b-l} as shown above must be nonzero too. For if $a_{u-b} = \bot$ and a_{b-l} nonzero, then $a_{b-l} + l$ would be an element of $\mathfrak{A}$ contained in b and larger than l, contradicting l maximal. Symmetrically, if $a_{b-l} = \bot$ and a_{u-b} nonzero, then $u - a_{u-b}$ would be an element of $\mathfrak{A}$ containing b and smaller than u, contradicting u minimal. This property — that nonzero elements of $\mathfrak{A}$ contained in the annulus $u - l$ depicted above decompose into disjoint unions

$a_{u-b} + a_{b-l}$ with a_{u-b}, a_{b-l} both nonzero — distinguishes elements of $\mathfrak{A}$ from b, since $b \wedge (u - b) = \bot$. The idea of the proof is to construct maps f, g making this difference visible, allowing to separate the element b from the subalgebra $\mathfrak{A}$.

To do this, we first formulate the above remarks in terms of boolean algebras. The sequence of inequalities $l < b < u$ gives a partition of unity $\{\neg u, u - b, b - l, l\}$, which gives a decomposition of $\mathfrak{B}$ as a simple product of principal ideals $\mathfrak{B} \cong \mathfrak{B}_{\neg u} \times \mathfrak{B}_{u-b} \times \mathfrak{B}_{b-l} \times \mathfrak{B}_l$. Since l and u are both in the subalgebra $\mathfrak{A}$, it decomposes similarly as $\mathfrak{A} \cong \mathfrak{A}_{\neg u} \times \mathfrak{A}_{u-l} \times \mathfrak{A}_l$. The inclusion $\mathfrak{A} \subseteq \mathfrak{B}$ decomposes correspondingly into three inclusions of principal ideals:

$$\mathfrak{A}_{\neg u} \subseteq \mathfrak{B}_{\neg u} \qquad \mathfrak{A}_{u-l} \subseteq \mathfrak{B}_{u-b} \times \mathfrak{B}_{b-l} \qquad \mathfrak{A}_l \subseteq \mathfrak{B}_l$$

Let i be the inclusion $\mathfrak{A}_{u-l} \subseteq \mathfrak{B}_{u-b} \times \mathfrak{B}_{b-l}$. As a homomorphism into a product of boolean algebras, i can be written uniquely as

$$i = (i_{u-b}, i_{b-l}) \quad \text{for some complete-boolean-algebra homomorphisms} \quad \begin{array}{l} i_{u-b} : \mathfrak{A}_{u-l} \to \mathfrak{B}_{u-b} \\ i_{b-l} : \mathfrak{A}_{u-l} \to \mathfrak{B}_{b-l} \end{array}$$

The earlier discussion shows that $i_{u-b}(a)$ and $i_{b-l}(a)$ are both nonzero for all nonzero $a \in \mathfrak{A}_{u-l}$. More is true: the homomorphisms i_{u-b}, i_{b-l} are injective. To see that i_{u-b} is injective, pick arbitrary $a, a' \in \mathfrak{A}_{u-l}$ with $i_{u-b}(a) = i_{u-b}(a')$. Unwinding definitions, this is equivalent to saying $a_{u-b} = a'_{u-b}$, where a decomposes as $a_{u-b} + a_{b-l}$ and a' as $a'_{u-b} + a'_{b-l}$. This forces $a_{b-l} = a'_{b-l}$ and hence $a = a'$ showing i_{u-b} injective, for otherwise $a - a' = (a_{u-b} + a_{b-l}) - (a'_{u-b} + a'_{b-l}) = a_{b-l} - a'_{b-l}$ would be an element of $\mathfrak{A}_{u-l}$ contained entirely in $b - l$, contradicting l maximal. An analogous argument shows i_{b-l} injective.

Since i_{u-b}, i_{b-l} are injective complete-boolean-algebra homomorphisms and therefere homomorphisms of standard measurable algebras, they correspond by Lemma B.26 to maps of standard enhanced measurable spaces with common codomain. By Lemma B.31, this cospan completes to a commutative square. Passing this square back through Lemma B.26 gives a standard measurable algebra $\mathfrak{C}_{u-l}$ and homomorphisms j, k fitting into the following commutative square:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "i_{u-b}"] \arrow[d, "i_{b-l}"'] & \mathfrak{B}_{u-b} \arrow[d, "j"] \\ \mathfrak{B}_{b-l} \arrow[r, "k"'] & \mathfrak{C}_{u-l} \end{tikzcd}$$

All ingredients needed to construct f, g are now at hand. Let $\mathfrak{C}$ be the standard measurable algebra $\mathfrak{B}_{\neg u} \times (\mathfrak{C}_{u-l} \mathbin{\hat{\otimes}} [0,1]) \times \mathfrak{B}_l$. Let p, q be the injective homomorphisms $\mathfrak{C}_{u-l} \times \mathfrak{C}_{u-l} \hookrightarrow \mathfrak{C}_{u-l} \mathbin{\hat{\otimes}} [0,1]$ defined by the following equations:

$$p(c_1, c_2) = (c_1 \mathbin{\hat{\otimes}} [0, 1/2]) + (c_2 \mathbin{\hat{\otimes}} [1/2, 1])$$
$$q(c_1, c_2) = (c_1 \mathbin{\hat{\otimes}} [0, 1/3]) + (c_2 \mathbin{\hat{\otimes}} [1/3, 1])$$

Let $f, g : \mathfrak{B} \hookrightarrow \mathfrak{C}$ be the following composites (note the only difference in their definitions is whether p or q is used at the end):

$$f = \left(\mathfrak{B} \xrightarrow{\sim} \mathfrak{B}_{\neg u} \times \mathfrak{B}_{u-b} \times \mathfrak{B}_{b-l} \times \mathfrak{B}_l \xhookrightarrow{1\times j\times k\times 1} \mathfrak{B}_{\neg u} \times (\mathfrak{C}_{u-l} \times \mathfrak{C}_{u-l}) \times \mathfrak{B}_l \xhookrightarrow{1\times p\times 1} \mathfrak{B}_{\neg u} \times (\mathfrak{C} \mathbin{\hat{\otimes}} [0,1]) \times \mathfrak{B}_l = \mathfrak{C}\right)$$
$$g = \left(\mathfrak{B} \xrightarrow{\sim} \mathfrak{B}_{\neg u} \times \mathfrak{B}_{u-b} \times \mathfrak{B}_{b-l} \times \mathfrak{B}_l \xhookrightarrow{1\times j\times k\times 1} \mathfrak{B}_{\neg u} \times (\mathfrak{C}_{u-l} \times \mathfrak{C}_{u-l}) \times \mathfrak{B}_l \xhookrightarrow{1\times q\times 1} \mathfrak{B}_{\neg u} \times (\mathfrak{C} \mathbin{\hat{\otimes}} [0,1]) \times \mathfrak{B}_l = \mathfrak{C}\right)$$

Since every element x of $\mathfrak{B}$ decomposes into a disjoint union $x_{\neg u} + x_{u-b} + x_{b-l} + x_l$ for some $x_{\neg u} \in \mathfrak{B}_{\neg u}$ and $x_{u-b} \in \mathfrak{B}_{u-b}$ and $x_{b-l} \in \mathfrak{B}_{b-l}$ and $x_l \in \mathfrak{B}_l$, the action of f and g on arbitrary elements of $\mathfrak{B}$ can be described by the following equations:

$$f(x_{\neg u} + x_{u-b} + x_{b-l} + x_l) = (x_{\neg u}, (j(x_{u-b}) \mathbin{\hat{\otimes}} [0, 1/2]) + (k(x_{b-l}) \mathbin{\hat{\otimes}} [1/2, 1]), x_l)$$
$$g(x_{\neg u} + x_{u-b} + x_{b-l} + x_l) = (x_{\neg u}, (j(x_{u-b}) \mathbin{\hat{\otimes}} [0, 1/3]) + (k(x_{b-l}) \mathbin{\hat{\otimes}} [1/3, 1]), x_l)$$

We now verify f, g have the desired property. First f, g agree on the subalgebra $\mathfrak{A}$: for any a in $\mathfrak{A}$, it holds that $j(a_{u-b}) = k(a_{b-l})$ because the square $ji_{u-b} = ki_{b-l}$ commutes, so

$$\begin{aligned} f(a_{\neg u} + a_{u-b} + a_{b-l} + a_l) &= (a_{\neg u}, (j(a_{u-b}) \mathbin{\hat{\otimes}} [0, 1/2]) + (k(a_{b-l}) \mathbin{\hat{\otimes}} [1/2, 1]), a_l) \\ &= (a_{\neg u}, (j(a_{u-b}) \mathbin{\hat{\otimes}} [0, 1/2]) + (j(a_{u-b}) \mathbin{\hat{\otimes}} [1/2, 1]), a_l) \\ &= (a_{\neg u}, j(a_{u-b}) \mathbin{\hat{\otimes}} ([0, 1/2] + [1/2, 1]), a_l) \\ &= (a_{\neg u}, j(a_{u-b}) \mathbin{\hat{\otimes}} \top, a_l) \\ &= (a_{\neg u}, j(a_{u-b}) \mathbin{\hat{\otimes}} ([0, 1/3] + [1/3, 1]), a_l) \\ &= (a_{\neg u}, (j(a_{u-b}) \mathbin{\hat{\otimes}} [0, 1/3]) + (j(a_{u-b}) \mathbin{\hat{\otimes}} [1/3, 1]), a_l) \\ &= (a_{\neg u}, (j(a_{u-b}) \mathbin{\hat{\otimes}} [0, 1/3]) + (k(a_{b-l}) \mathbin{\hat{\otimes}} [1/3, 1]), a_l) \\ &= g(a_{\neg u} + a_{u-b} + a_{b-l} + a_l). \end{aligned}$$

Second, $f(b) \notin \operatorname{im} g$: every element in the image of g is of the form

$$(x_{\neg u}, (j(x_{u-b}) \hat{\otimes} [0, 1/3]) + (k(x_{b-l}) \hat{\otimes} [1/3, 1]), x_l)$$

but

$$\begin{aligned} f(b) &= f(\bot + \bot + (b - l) + l) \\ &= (\bot, (j(\bot) \hat{\otimes} [0, 1/2]) + (k(\top_{\mathfrak{B}_{b-l}}) \hat{\otimes} [1/2, 1]), \top_{\mathfrak{B}_l}) \\ &= (\bot, (\bot \hat{\otimes} [0, 1/2]) + (\top \hat{\otimes} [1/2, 1]), \top) \\ &= (\bot, \top \hat{\otimes} [1/2, 1], \top) \end{aligned}$$

and there are no x_{u-b}, x_{b-l} such that $(j(x_{u-b}) \hat{\otimes} [0, 1/3]) + (k(x_{b-l}) \hat{\otimes} [1/3, 1]) = \top \hat{\otimes} [1/2, 1]$. □

C GENERAL SHEAF THEORY

C.1 Atomic sheaves

Notation C.1. If F is a presheaf on C and $f : X \to Y$ a morphism in C and $y \in FY$, we will write $y \cdot f$ for the element $F(f)(y) \in FX$.

Definition C.2 (f-invariance)**.** Let F be a presheaf on C, $y \in Fd$ an element of F, and $f : d \to c$ a morphism in C. Following Simpson [47], say y is *f-invariant* if for all $g, h : e \to d$ with $fg = fh$ it holds that $y \cdot g = y \cdot h$. The following diagram illustrates the situation:

$$y \cdot g = y \cdot h \in Fe \longleftarrow\!\shortmid\; y \in Fd$$

$$\begin{tikzcd} e \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & d \arrow[r, "f"] & c \end{tikzcd}$$

Fact C.3. The atomic topology exists for C iff C satisfies the *right Ore property*: for any two morphisms $f : c \to e$ and $g : d \to e$, there exists an object b and morphisms $h : b \to c$ and $k : b \to d$ such that

$$\begin{tikzcd} b \arrow[r, "k"] \arrow[d, "h"'] & d \arrow[d, "g"] \\ c \arrow[r, "f"'] & e \end{tikzcd}$$

commutes [30, Example III.2(f), p.115].

Definition C.4 (atomic sheaf)**.** Let C be a category for which the atomic topology exists. A presheaf F on C is an *atomic sheaf* if and only if it satisfies the following condition [30, Lemma III.4.2]: for all morphisms $f : d \to c$ in C, the function Ff is an inclusion $Fc \hookrightarrow Fd$ whose image is the subset of f-invariant elements of Fd. More explicitly: for all morphisms $f : d \to c$ and f-invariant elements $y \in Fd$, there exists a unique $x \in Fc$ with $y = x \cdot f$.

C.2 Continuous group-invariant sets

Definition C.5 (category of G-sets)**.** For G a topological group, the *category of continuous G-sets*, written $G\,\mathbf{Set}$, is the category whose objects are sets X equipped with a continuous right action $(\cdot_X) : X \times G \to X$ (where X is given the discrete topology) and whose morphisms from X to Y are functions $f : X \to Y$ that are equivariant: $f(x \cdot_X g) = f(x) \cdot_Y g$ for all $x \in X$ and $g \in G$.

Lemma C.6. For any topological group G, there is an isomorphism of categories $G\,\mathbf{Set} \cong G^{\mathrm{op}}\,\mathbf{Set}$.

Proof. Whereas objects of $G\,\mathbf{Set}$ are sets X equipped with a continuous right action $(\cdot) : X \times G \to X$, objects of $G^{\mathrm{op}}\,\mathbf{Set}$ are sets X equipped with a continuous left action $(\cdot^{\mathrm{op}}) : G \times X \to X$. Every $(X, \cdot) \in G\,\mathbf{Set}$ corresponds to $(X, \cdot^{\mathrm{op}}) \in G^{\mathrm{op}}\,\mathbf{Set}$ by setting $g \cdot^{\mathrm{op}} x := x \cdot g^{-1}$. (This is indeed a left action, since $g \cdot^{\mathrm{op}} h \cdot^{\mathrm{op}} x = x \cdot h^{-1} \cdot g^{-1} = x \cdot (gh)^{-1} = gh \cdot^{\mathrm{op}} x$, and it is continuous because $(\cdot)$ and $(\cdot^{\mathrm{op}})$ yield the same stabilizer subgroups.) A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ in $G\,\mathbf{Set}$ is a function $f : X \to Y$ satisfying $f(x \cdot_X g) = f(x) \cdot_Y g$ for all $g \in G$, which is equivalent to $f(x \cdot_X g^{-1}) = f(x) \cdot_Y g^{-1}$ for all $g \in G$, making f also a morphism $(X, \cdot^{\mathrm{op}}_X) \to (Y, \cdot^{\mathrm{op}}_Y)$ in $G^{\mathrm{op}}\,\mathbf{Set}$. □

C.3 Presheaves with minimal supports

Definition C.7 (category with minimal supports)**.** A category C *has minimal supports* if every coslice c/C has a terminal object. For any object c of C, call the terminal object $(c^*, p : c \to c^*)$ of c/C the *support* of c. An object c has *trivial support* or is *trivially-supported* if its support is the identity map $1 : c \to c$.

Lemma C.8. If C has minimal supports and c has support $p : c \to c^*$ and $f : c \to d$, then d has support $!_f : d \to c^*$ where $!_f$ is the unique morphism $f \to p$ in c/C.

Proof. Consider the following diagram.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "p"] & \\ d \arrow[rr, dashed, "!_f"'] \arrow[d, "g"'] & & c^* \\ e \arrow[urr, dashed, "!_{gf}"'] & & \end{tikzcd}$$

The upper triangle depicts the situation given: d has support p, and, because p terminal in c/C, there exists a unique map $!_f$ making the upper triangle commute in C. To see that the map $!_f$ is the terminal object of d/C, fix an arbitrary object g of d/C as shown. Because p terminal in c/C, there exists a unique $!_{gf}$ such that the outer quadrilateral commutes. The composite $!_{gf}g$ is just as good as $!_f$ when it comes to making the upper triangle commute, so uniqueness of $!_f$ implies the lower triangle commutes. Uniqueness of $!_{gf}$ in making the lower triangle commute then follows from its uniqueness in making the quadrilateral commute. □

Lemma C.9. If C has minimal supports and c has support $p : c \to c^*$, then c^* has support $1 : c^* \to c^*$. Thus every object of C has a map to an object with trivial support.

Proof. Set $f = p$ in Lemma C.8; have $!_p = 1_{c^*}$ by uniqueness of $!_p$ in making the triangle $!_p p = p$ commute. □

Lemma C.10. If C has minimal supports and c has support $p : c \to c^*$ and $f : d \to c$, then d has support $pf : d \to c^*$. In particular, if c has trivial support then any map $f : d \to c$ supports d.

Proof. Consider the following diagram.

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "q"] & \\ c \arrow[rr, dashed, "!_f"] \arrow[dr, "p"'] & & d^* \arrow[dl, dashed, bend left] \\ & c^* \arrow[ur, dashed, bend left] & \end{tikzcd}$$

The solid arrows depict the situation given: c has support p and $f : d \to c$, and, since C supported, d has some support $q : d \to d^*$. To show pf supports d, it suffices to show $pf \cong q$ as objects of d/C. By Lemma C.8, the unique map $!_f$ making the upper triangle commute is a support for c. Thus $!_f$ and p are both supports for c; thus $!_f \cong p$ as objects of c/C, and there exists the unlabelled dashed isomorphism making the lower triangle commute. Since both the upper and lower triangles commute, the whole diagram commutes, and the dashed isomorphism gives $f \cong q$ in d/C as desired. □

Lemma C.11. If C has minimal supports and contains only epis, and c has trivial support, then every $f : c \to d$ is an isomorphism.

Proof. Since $1 : c \to c$ terminal in c/C, there is a unique $g : d \to c$ with $gf = 1$, so f epi and a left inverse, so f iso. □

Definition C.12 (presheaf with minimal supports)**.** Say a presheaf F on C *has minimal supports* if its category of elements $\mathrm{El}(F)$ is a category with minimal supports. Unwinding definitions, F has minimal supports if for every $x : Fc$ there exists $x^* : Fc^*$ and $p : c \to c^*$ with $x = x^* \cdot p$ such that for all $x' : Fc'$ and $p' : c \to c'$ with $x = x' \cdot p$, there exists a unique $q : c' \to c^*$ with $x' = x^* \cdot q$ and $p = qp'$. As a terminal object, (c^*, x^*, p) is unique up to unique isomorphism, which in this case means unique mod $(c^*, x^*, p) \sim (c^\sharp, x^* \cdot i, i^{-1}p)$ for all isomorphisms $i : c^\sharp \to c^*$.

Intuition C.13. For sheaves on a category of measurable spaces, a sheaf F is supported if every element $x : F\Omega$ can be expressed in terms of a "smallest sample space" Ω^*. This sample space is not necessarily unique—any two-point space will do for modelling a boolean random variable, for example—but is unique up to unique isomorphism of measurable spaces.

Note C.14. This notion of support is related to the one presented in Staton [50, Section 4.4.1]: a sheaf with minimal supports as defined here corresponds, under the terminology there, to a sheaf for which every element has a least support.

Lemma C.15. For any category C and object c of C, the representable presheaf $よc$ has minimal supports.

Proof. Fix $f : C(x, c)$. The morphism $(f : C(x, c)) \xrightarrow{f} (1 : C(c, c))$ is terminal in $f/\mathrm{El}(よc)$. To see this, fix arbitrary $(f : C(x, c)) \xrightarrow{p} (g : C(y, c))$ with aim to find a unique dashed morphism making the following triangle commute in $f/\mathrm{El}(よc)$:

$$\begin{tikzcd} & f : C(x,c) \arrow[dl, "p"'] \arrow[dr, "f"] & \\ g : C(y,c) \arrow[rr, dashed, "!_p"'] & & 1 : C(c,c) \end{tikzcd}$$

Such a dashed arrow $!_p$ must satisfy $1 \circ !_p = g$, forcing $!_p = g$. It only remains to check $!_p$ makes the triangle commute; this follows from the fact that p is a morphism from f to g in $\mathrm{El}(\text{よ}c)$. ◻

Definition C.16. For $U : D \to C$ a functor and G an essentially small groupoid in D, define $\text{よ}(UG)$ to be the presheaf $\mathrm{colim}_{c \in G}\, \text{よ}(Uc)$, where よ is the Yoneda embedding. More explicitly, the action of $\text{よ}(UG)$ on objects is

$$\text{よ}(UG)(c) = \{\text{morphisms } f : c \to Ud \text{ for } d \text{ in } G\}/\!\sim$$

where $(f : c \to Ud) \sim (g : c \to Ue)$ if and only if $U(\pi)f = g$ for some $\pi : d \to e$ in G, and the action on morphisms is

$$\text{よ}(UG)(f : c \to d) = \begin{pmatrix} \text{よ}(UG)(d) \to \text{よ}(UG)(c) \\ [g : d \to Ue] \mapsto [gf : c \to Ue] \end{pmatrix}.$$

Definition C.17 (representables modulo a groupoid)**.** For C any category and G a small groupoid in C, let $\text{よ}G$ be the *presheaf of representables modulo* G. This is a specialization of Definition C.16 to the case $U = 1_C$.

Lemma C.18. For $U : D \to C$ with C a category of epis and G a groupoid in D, the presheaf $\text{よ}(UG)$ has minimal supports.

Proof. The proof is similar to Lemma C.15. Fix an equivalence class $[f : C(x, Uc)]$. The morphism $[f : C(x, Uc)] \xrightarrow{f} [1 : C(Uc, Uc)]$ is terminal in $[f]/\mathrm{El}(\text{よ}(UG))$. To see this, fix arbitrary $p : x \to y$ and $[f : C(x, Uc)] \xrightarrow{p} [g : C(y, Ud)]$ with aim to find a unique dashed morphism making the following triangle commute in $[f]/\mathrm{El}(\text{よ}(UG))$:

$$\begin{tikzcd} & {[f : C(x, Uc)]} \arrow[dl, "p"'] \arrow[dr, "f"] & \\ {[g : C(y, Ud)]} \arrow[rr, dashed, "!_p"'] & & {[1 : C(Uc, Uc)]} \end{tikzcd}$$

Unwinding definitions, this diagram says $[g] \cdot p = [gp] = [f]$, so $U(\pi)gp = f$ for some $\pi : d \to c$ in G. Commutativity of the above triangle requires solving $!_p p = f$ for $!_p$. Since p epi and $U(\pi)gp = f$, the composite $U(\pi)g$ is the only possible solution. It only remains to check that setting $!_p := U(\pi)g$ gives a morphism in $\mathrm{El}(\text{よ}(UG))$ from $[g]$ to $[1]$, and indeed $[1] \cdot !_p = [1] \cdot U(\pi)g = [U(\pi)g] = [g]$. ◻

C.4 The Day convolution of sheaves with minimal supports

In this section we describe conditions under which Day convolution preserves atomic sheaves.

Definition C.19 (tensor product of presheaves)**.** Let $(C, \otimes, \mathrm{I})$ be a symmetric monoidal category. Given two presheaves P, Q on C, their *Day convolution* [14] is defined by the following coend:

$$(P \otimes Q)c = \int^{c_P, c_Q \in C} Pc_P \times Qc_Q \times C(c, c_P \otimes c_Q)$$

This product makes the category $\mathrm{PSh}(C)$ of presheaves on C into a symmetric monoidal category, with unit $\text{よ}\mathrm{I}$ where よ is the Yoneda embedding.

The following lemma gives a concrete representation for the Day convolution of two atomic sheaves in the special case where the sheaves have minimal supports and the base category is made only of epis.

Lemma C.20. Let $(C, \otimes, \mathrm{I})$ be a symmetric monoidal category of epis for which the atomic topology exists. If F and G are atomic sheaves on C with minimal supports, then $iF \otimes iG$ is the presheaf

$$(iF \otimes iG)(c) = \left(\begin{array}{l} \text{tuples } (c_x, c_y, f : c \to c_x \otimes c_y, x : Fc_x, y : Gc_y), \text{ abbreviated } (f, x : Fc_x, y : Gc_y), \\ \text{where } x \text{ and } y \text{ have trivial support, mod the equivalence relation} \\ \quad (f, x : Fc_x, y : Gc_y) \sim (g, a : Fc_a, b : Gc_b) \\ \text{iff there exist isos } h : c_a \to c_x \text{ and } k : c_b \to c_y \\ \text{such that } f = (h \otimes k)g \text{ and } a = x \cdot h \text{ and } b = y \cdot k \end{array} \right)$$

Proof. The Day convolution $iF \otimes iG$ is a presheaf that sends c to

$$\begin{aligned} (iF \otimes iG)(c) &= \int^{c_x, c_y} C(c, c_x \otimes c_y) \times Fc_x \times Gc_y \\ &= \left(\begin{array}{l} \text{tuples } (c_x, c_y, f : c \to c_x \otimes c_y, x : Fc_x, y : Gc_y), \text{ abbreviated } (f, x : Fc_x, y : Gc_y), \\ \text{mod the equivalence relation generated by} \\ \quad ((g \otimes h) \circ f, x : Fc_x, y : Gc_y) \sim (f, x \cdot g : Fc'_x, y \cdot h : Gc'_y) \\ \text{for all } f : c \to c'_x \otimes c'_y \text{ and } g : c'_x \to c_x \text{ and } h : c'_y \to c_y \end{array} \right) \end{aligned}$$

Since F and G have minimal supports, every element $[f, x : Fc_x, y : Gc_y]$ of $(iF \otimes iG)(c)$ (writing $[-]$ for equivalence class) is of the form $[f, x^* \cdot p_x : Fc_x^*, y^* \cdot p_y : Gc_y^*]$ for some (x^*, p_x) supporting x and (y^*, p_y) supporting y. This simplifies things: the elements x^* and y^* have support $1_{c_x^*}$ and $1_{c_y^*}$ by Lemma C.9, and $[f, x^* \cdot p_x, y^* \cdot p_y] = [(p_x \otimes p_y) \circ f, x^*, y^*]$, so every element of $(iF \otimes iG)(c)$ is of the form $[f, x : Fc_x, y : Gc_y]$ for some x and y whose supports are the identity maps 1_{c_x} and 1_{c_y} respectively:

$$(iF \otimes iG)(c) = \begin{pmatrix} \text{trivially-supported tuples } (f, x : Fc_x, y : Gc_y) \\ \text{mod the equivalence relation generated by } (\sim) \text{ as above} \end{pmatrix}$$

Suppose two trivially-supported tuples $(f, x : Fc_x, y : Gc_y)$ and $(g, a : Fc_a, b : Gc_b)$ are related by $(\sim)$, so there exist $h : c_a \to c_x$ and $k : c_b \to c_y$ such that

$$f = (h \otimes k)g \qquad \text{and} \qquad a = x \cdot h \qquad \text{and} \qquad b = y \cdot k.$$

The equation $a = x \cdot h$ corresponds to a morphism $(a : Fc_a) \xrightarrow{h} (x : Fc_x)$ in $\mathrm{El}(F)$. Now a has support 1_{c_a} by assumption and $\mathrm{El}(F)$ contains only epis because C does by hypothesis, so h iso in $\mathrm{El}(F)$ by Lemma C.11. This implies h iso in C. Running the same argument on the equation $b = y \cdot k$ gives k iso in C. Thus

$$(f, x, y) \sim (g, a, b) \iff \text{there exist } h, k \textbf{ iso} \text{ such that } f = (h \otimes k)g \text{ and } a = x \cdot h \text{ and } b = y \cdot k. \tag{2}$$

This is an equivalence relation on trivially-supported tuples:

- Reflexivity: choose $h = k = 1$.
- Symmetry: if $f = (h \otimes k)g$ and $a = x \cdot h$ and $b = y \cdot k$, then $(h^{-1} \otimes k^{-1})f = g$ and $a \cdot h^{-1} = x$ and $b \cdot k^{-1} = y$, so if h, k witness $(f, x, y) \sim (g, a, b)$ then h^{-1}, k^{-1} witness $(g, a, b) \sim (f, x, y)$.
- Transitivity: suppose $(f, x, y) \sim_{p,q} (g, a, b) \sim_{r,s} (h, u, v)$, where the subscripts on $\sim$ indicate the witnesses for the given relation. This gives

$$\begin{aligned} f &= (p \otimes q)g = (p \otimes q)(r \otimes s)h = (pr \otimes qs)h \\ u &= a \cdot r = x \cdot p \cdot r = x \cdot pr \\ v &= b \cdot s = y \cdot q \cdot s = y \cdot qs \end{aligned}$$

  which together says $(f, x, y) \sim_{pr,qs} (h, u, v)$.

Thus the quotienting done by the coend in $(iF \otimes iG)(c)$ is precisely a quotient by $(\sim)$ on trivially supported tuples, as claimed:

$$(iF \otimes iG)(c) = \big(\text{trivially-supported tuples } (f, x : Fc_x, y : Gc_y)\big) \; /\!\sim$$

□

Definition C.21 (semicartesian monoidal category)**.** A monoidal category $(C, \otimes, \mathrm{I})$ is *semicartesian* if I is the terminal object of C. This implies the existence of projection maps $\mathrm{fst} : a \otimes b \to a$ and $\mathrm{snd} : a \otimes b \to b$, defined by the composites

$$\mathrm{fst} = \left(a \otimes b \xrightarrow{1 \otimes !} a \otimes 1 \cong a \otimes \mathrm{I} \cong a \right)$$

$$\mathrm{snd} = \left(a \otimes b \xrightarrow{! \otimes 1} 1 \otimes b \cong \mathrm{I} \otimes b \cong a \right)$$

where every occurrence of ! denotes the unique morphism into the terminal object.

Definition C.22 (category of supports)**.** Call a symmetric semicartesian monoidal category $(C, \otimes, \mathrm{I})$ a *category of supports* if

- Every map in C is epi;
- The two projection maps fst, snd are jointly monic: two maps $f, g : c \to d \otimes e$ are equal iff $\mathrm{fst}\, f = \mathrm{fst}\, g$ and $\mathrm{snd}\, f = \mathrm{snd}\, g$;
- The atomic topology exists for C;
- For every groupoid G in C, the presheaf of representables modulo G is an atomic sheaf.

Lemma C.23. Let C be a category of supports and F, G atomic sheaves on C with minimal supports. The Day convolution for presheaves $iF \otimes iG$ is an atomic sheaf with minimal supports.

Proof. We use the concrete representation for $iF \otimes iG$ calculated in Lemma C.20.

- $iF \otimes iG$ is a sheaf: Fix $p : c' \to c$ and an equivalence class $[f, x : Fc_x, y : Gc_y] : (iF \otimes iG)(c')$ of trivially-supported tuples that is p-invariant, so for all $q, r : c'' \to c'$ satisfying $pq = pr$ it holds that $[f, x, y] \cdot q = [f, x, y] \cdot r$. We are done if we can show that there exists a unique extension of $[f, x, y]$ to an element $[g, a, b]$ of $(iF \otimes iG)(c)$ such that $[g, a, b] \cdot p = [f, x, y]$.
  For any q and r, we have $[f, x, y] \cdot q = [fq, x, y] = [fr, x, y] = [f, x, y] \cdot r$ iff $(fq, x, y) \sim (fr, x, y)$, iff there exist isos h, k with

$$fq = (h \otimes k)fr \text{ and } x = x \cdot h \text{ and } y = y \cdot k.$$

  Thus p-invariance of $[f, x, y]$ says that for all q, r with $pq = pr$ it holds that $fq = (h \otimes k)fr$ for some isos $h : c_x \to c_x$ and $k : c_y \to c_y$. This is precisely what it means for $[f]_G$ to be p-invariant as an element of $C(c', G)$, where G is the groupoid of isos of the form $h \otimes k$.

(Note h and k happen to be automorphisms here, but G also contains non-automorphisms. This will be relevant later.) The presheaf $C(-,G)$ is an atomic sheaf because C is a category of supports, so there exist c'_x and c'_y and $\overline{f} : c \to c'_x \otimes c'_y$ with $\overline{f}p \sim_G f$, unique up to $(\sim_G)$. Existence of $(c'_x, c'_y, \overline{f})$ is equivalent to the existence of a map $\overline{f} : c \to c_x \otimes c_y$ with $\overline{f}p = f$: any tuple $(c'_x, c'_y, \overline{f})$ with $\overline{f}p = (h \otimes k)f$ for some h, k in G yields a map $(h^{-1} \otimes k^{-1})\overline{f}$ with $(h^{-1} \otimes k^{-1})\overline{f}p = f$, and conversely if $\overline{f}p = f$ then one has a tuple $(c_x, c_y, \overline{f})$ with $\overline{f}p \sim_G f$.

The equivalence class $[\overline{f}, x, y]$ is an element of $(iF \otimes iG)(c)$ satisfying $[\overline{f}, x, y] \cdot p = [f, x, y]$. It only remains to show that it is the unique such. Suppose $[g, a, b] \cdot p = [gp, a, b] = [f, x, y]$, so $(f, x, y) \sim (gp, a, b)$, so there exist isos h, k with

$$f = (h \otimes k)gp \qquad \text{and} \qquad a = x \cdot h \qquad \text{and} \qquad b = y \cdot k.$$

Then $(h \otimes k)gp = f = \overline{f}p$, so $(h \otimes k)g = \overline{f}$ by p epi, so

$$[g, a, b] = [g, x \cdot h, y \cdot k] = [(h \otimes k)g, x, y] = [\overline{f}, x, y]$$

establishing uniqueness of $[\overline{f}, x, y]$.

- $iF \otimes iG$ is has minimal supports: fix $[f, x : Fc_x, y : Gc_y] : (iF \otimes iG)(c)$. We have $[f, x, y] = [1_{c_x \otimes c_y}, x, y] \cdot f$, giving an object $[f, x, y] \xrightarrow{f} [1, x, y]$ of $[f, x, y]/\mathrm{El}(iF \otimes iG)$. This object is terminal: fix arbitrary $[f, x, y] \xrightarrow{p} [g, a, b]$ with aim to find $!_p$ making

$$\begin{tikzcd} & {[f,x,y]} \arrow[dl, "p"'] \arrow[dr, "f"] & \\ {[g,a,b]} \arrow[rr, dashed, "!_p"'] & & {[1,x,y]} \end{tikzcd}$$

commute. Since C is a category of epis and p a morphism in C, any dashed morphism completing this triangle must be unique, so it only remains to find one such. Unpacking the arrow $[f, x, y] \xrightarrow{p} [g, a, b]$ gives the equations

$$f = (h \otimes k)gp \qquad a = x \cdot h \qquad b = y \cdot k$$

Commutativity of the triangle requires $!_p p = f$, which suggests setting $!_p = (h \otimes k)g$. It only remains to check that $[1, x, y] \cdot !_p = [g, a, b]$. Indeed, $[1, x, y] \cdot !_p = [(h \otimes k)g, x, y] = [g, x \cdot h, y \cdot k] = [g, a, b]$.

□

Lemma C.24. Let C be a category of supports and F, G atomic sheaves on C with minimal supports. The natural transformation

$$i : F \otimes G \hookrightarrow F \times G$$
$$i_c[f, x : Fc_x, y : Gc_y] = (x \cdot \mathrm{fst}\, f : Fc,\ y \cdot \mathrm{snd}\, f : Gc)$$

is a monic map of sheaves, making $F \otimes G$ a subobject of $F \times G$.

Proof. Because F and G have minimal supports, the sheaf tensor product $F \otimes G$ coincides with the presheaf tensor product (Lemma C.23). The map i is defined above on trivially-supported tuples; it respects the equivalence relation because

$$i_c[f, x \cdot h, y \cdot k] = (x \cdot h \cdot \mathrm{fst}\, f,\ y \cdot k \cdot \mathrm{snd}\, f) = (x \cdot \mathrm{fst}(h \otimes k)f,\ y \cdot \mathrm{snd}(h \otimes k)f] = i_c[(h \otimes k)f, x, y].$$

It is a natural transformation:

$$i_c[fp, x, y] = (x \cdot \mathrm{fst}\, fp, y \cdot \mathrm{snd}\, fp) = (x \cdot \mathrm{fst}\, f, y \cdot \mathrm{snd}\, f) \cdot p = i_c[f, x, y] \cdot p$$

Finally, each component of i is monic. Fix arbitrary trivially-supported $[f, x, y]$ and $[g, a, b]$ and suppose $i_c[f, x, y] = i_c[g, a, b]$, so $x \cdot \mathrm{fst}\, f = a \cdot \mathrm{fst}\, g$ and $y \cdot \mathrm{snd}\, f = b \cdot \mathrm{snd}\, g$. This corresponds to the following diagrams in $\mathrm{El}(F)$ and $\mathrm{El}(G)$ respectively:

$$\begin{tikzcd} & {x \cdot \mathrm{fst}\, f = a \cdot \mathrm{fst}\, g} \arrow[dl, "\mathrm{fst}\, f"'] \arrow[dr, "\mathrm{fst}\, g"] & \\ x \arrow[rr, dashed, bend left, "h"] & & a \arrow[ll, dashed, bend left, "h^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} & {y \cdot \mathrm{snd}\, f = b \cdot \mathrm{snd}\, g} \arrow[dl, "\mathrm{snd}\, f"'] \arrow[dr, "\mathrm{snd}\, g"] & \\ y \arrow[rr, dashed, bend left, "k"] & & b \arrow[ll, dashed, bend left, "k^{-1}"] \end{tikzcd}$$

The solid arrows depict the situation given. Since both x and a have trivial support, the common value $x \cdot \mathrm{fst}\, f = a \cdot \mathrm{fst}\, g$ has both $\mathrm{fst}\, f$ and $\mathrm{fst}\, g$ as supports by Lemma C.10. Any two supports for the same object are isomorphic, so $\mathrm{fst}\, f \cong \mathrm{snd}\, g$ in the slice category $(x \cdot \mathrm{fst}\, f)/\mathrm{El}(F)$, giving h, h^{-1} making the triangle on the left commute. Analogously, $\mathrm{snd}\, f$ and $\mathrm{snd}\, g$ are both supports for the common value $y \cdot \mathrm{snd}\, f = b \cdot \mathrm{snd}\, g$, giving k, k^{-1} making the triangle on the right commute. Unpacking what it means for h and k to be morphisms in $\mathrm{El}(F)$ and $\mathrm{El}(G)$ respectively gives $a = x \cdot h$ and $b = y \cdot k$, so to get $[f, x, y] = [g, a, b]$ it only remains to show $f = (h \otimes k)g$. Two maps $c \to c_x \otimes c_y$ are equal iff they are equal when postcomposed with the projections fst and snd, and indeed

$$\mathsf{fst}(h \otimes k)g = h\,\mathsf{fst}\,g \overset{(*)}{=} \mathsf{fst}\,f \qquad \text{and} \qquad \mathsf{snd}(h \otimes k)g = k\,\mathsf{snd}\,g \overset{(*)}{=} \mathsf{snd}\,f$$

where the equations marked $(*)$ follow from commutativity of the triangles above. □

C.5 Nominal situations

Definition C.25 (nominal situation)**.** A *nominal situation* is a tuple $(C, C_\infty, \mathrm{c}_\infty : C_\infty, (i_c : c \hookrightarrow \mathrm{c}_\infty)_{c:C}, G)$ where

- C is a full subcategory of C_∞
- C and C_∞ consist only of monic maps
- The atomic topology exists for C^{op}
- G is a subgroup of $\mathrm{Aut}(\mathrm{c}_\infty)$
- (Closure) The special monos i_c hit every subobject of the form πi_c in c_∞ with $\pi \in G$. That is, for every c and auto $\pi \in G$ there exists an isomorphism $f : c \xrightarrow{\sim} c'$ in C with $\pi i_c = i_{c'} f$:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, dashed, "f"'] & c' \arrow[u, hook, "i_{c'}"'] \end{tikzcd}$$

- (Homogeneity) For every map $f : c \hookrightarrow d$ in C there exists $\pi \in G$ with $\pi i_c = i_d f$:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, dashed, "\pi"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, hook, "f"'] & d \arrow[u, hook, "i_d"'] \end{tikzcd}$$

- (Correspondence) The map
$$\mathrm{Fix}\, i := \{\pi \mid \pi i = i\} \subseteq G$$
that sends every mono $i : c \hookrightarrow \mathrm{c}_\infty$ to the subgroup of G fixing it gives an contravariant equivalence between subobjects of c_∞ and subgroups of G fixing those subobjects. (This mapping is automatically faithful because its domain — the subobjects of c_∞ — is a thin category. The nontrivial part is the requirement that Fix be full, which is to say that if $\mathrm{Fix}\, i \subseteq \mathrm{Fix}\, j$ then j factors through i, so that the triangle
$$\begin{tikzcd} & \mathrm{c}_\infty & \\ \mathrm{dom}\, j \arrow[ur, hook, "j"] \arrow[rr, dashed] & & \mathrm{dom}\, i \arrow[ul, hook', "i"'] \end{tikzcd}$$
commutes.)
- (Cofinality) For every finite family of objects $(c_j)_{j \in J}$ in C there exists an object c^* in C with $\mathrm{Fix}\, i_{c^*} \subseteq \bigcap_j \mathrm{Fix}\, i_{c_j}$.

Definition C.26. In a nominal situation $(C, C_\infty, \mathrm{c}_\infty, i_\bullet, G)$, say an automorphism $\pi \in G$ *refines* a map $f : c \to d$ if the following square commutes:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, hook, "f"'] & d \arrow[u, hook, "i_d"'] \end{tikzcd}$$

In this language, Homogeneity says every map is refined by some automorphism.

Lemma C.27. Refinement respects composition: if π_f refines $f : c \to d$ and π_g refines $g : d \to e$, then $\pi_g \pi_f$ refines gf, and if π refines an iso f then π^{-1} refines f^{-1}.

Proof. First, if π_f refines f and π_g refines g then pasting the respective commutative squares together gives the following commutative rectangle witnessing $\pi_g \pi_f$ refines gf:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi_f"] & \mathrm{c}_\infty \arrow[r, "\pi_g"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, "f"'] & d \arrow[u, hook, "i_d"] \arrow[r, "g"'] & e \arrow[u, hook, "i_e"] \end{tikzcd}$$

Second, if π refines $f : c \to d$ iso then $\pi i_c = i_d f$, so $i_c f^{-1} = \pi^{-1} i_d$, so π^{-1} refines f^{-1}. □

Definition C.28 (refinement topology)**.** Let $(C, C_\infty, \mathrm{c}_\infty, \boldsymbol{i}_\bullet, G)$ be a nominal situation. For any iso $f : c \xrightarrow{\sim} d$ in C, let $\mathcal{R}_f$ be the collection of all automorphisms in G that refine f:

$$\mathcal{R}_f := \{\pi : \mathrm{c}_\infty \xrightarrow{\sim} \mathrm{c}_\infty \mid \pi i_c = i_d f\}$$

The *refinement topology* is the topology on G consisting of unions of finite intersections of sets $\mathcal{R}_f$ for isos f of C.

Lemma C.29. *In a nominal situation $(C, C_\infty, \mathrm{c}_\infty, \boldsymbol{i}_\bullet, G)$, the group G is continuous for the refinement topology.*

Proof. We show that the inverse and multiplication maps are continuous.

- Inverse is continuous: for all isos f of C,

$$\mathrm{inverse}^{-1}(\mathcal{R}_f) = \{\pi \mid \pi^{-1} \in \mathcal{R}_f\} = \{\pi \mid \pi^{-1} \text{ refines } f\} = \{\pi \mid \pi \text{ refines } f^{-1}\} = \{\pi \mid \pi \in \mathcal{R}_{f^{-1}}\} = \mathcal{R}_{f^{-1}}$$

  so the preimage of every basis element is open.
- Multiplication is continuous: fix an iso f of C and suppose $(\sigma, \pi) \in \mathrm{mul}^{-1}(\mathcal{R}_f)$ with aim to find an open neighborhood $O_\sigma \times O_\pi$ such that $(\sigma, \pi) \in O_\sigma \times O_\pi \subseteq \mathrm{mul}^{-1}(\mathcal{R}_f)$. By assumption $\sigma\pi$ refines f, so

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \arrow[r, "\sigma"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[rr, "f"'] & & d \arrow[u, hook, "i_d"'] \end{tikzcd}$$

  commutes. By Closure, the subobject πi_c is isomorphic to $i_{c'}$ for some c'. That is, there exists an object c' and isomorphism $g : c \to c'$ such that

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, "g"'] & c' \arrow[u, hook, "i_{c'}"] \end{tikzcd}$$

  commutes. Combining this square with the rectangle above gives

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \arrow[r, "\sigma"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[rr, "f"] \arrow[dr, "g"'] & & d \arrow[u, hook, "i_d"'] \\ & c' \arrow[uu, hook, bend left, "i_{c'}"] \arrow[ur, dashed, "fg^{-1}"'] & \end{tikzcd}$$

  where the back rectangle and the left quadrilateral commute. Since g iso, the dashed arrow exists and makes the lower triangle commute. By diagram chase, the right quadrilateral commutes when precomposed with g; since g iso, this implies the right quadrilateral commutes. This gives the following commutative rectangle:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \arrow[r, "\sigma"] & \mathrm{c}_\infty \\ c \arrow[u, hook, "i_c"] \arrow[r, "g"'] & c' \arrow[u, hook, "i_{c'}"] \arrow[r, "fg^{-1}"'] & d \arrow[u, hook, "i_d"] \end{tikzcd}$$

  Translating this rectangle into words, we have that π refines g and σ refines fg^{-1} and and $\sigma\pi$ refines f. But note that for any other π' and σ' we would still have $\sigma'\pi'$ refining f so long as π' refines g and σ' refines fg^{-1}. In other words,

$$(\sigma, \pi) \in \mathcal{R}_{fg^{-1}} \times \mathcal{R}_g \subseteq \mathrm{mul}^{-1}(\mathcal{R}_f)$$

  and we have found a suitable open neighborhood as required.

□

Definition C.30. In a nominal situation $(C, C_\infty, \mathrm{c}_\infty, i_\bullet, G)$, let $\mathbf{Fix}_C$ be the category whose objects are subgroups $\mathrm{Fix}\, i_c$ for all c in C and whose morphisms $\mathrm{Fix}\, i_c \to \mathrm{Fix}\, i_d$ are cosets $(\mathrm{Fix}\, i_d)\pi$ such that $\pi g \pi^{-1}$ fixes i_d for all $g \in G$ that fix i_c, with composition $\mathrm{Fix}\, i_c \xrightarrow{(\mathrm{Fix}\, i_d)\pi} \mathrm{Fix}\, i_d \xrightarrow{(\mathrm{Fix}\, i_e)\sigma} \mathrm{Fix}\, i_e$ given by $(\mathrm{Fix}\, i_e)\sigma\pi$.

Lemma C.31. *For any nominal situation $(C, C_\infty, \mathrm{c}_\infty, \boldsymbol{i}_\bullet, G)$, there is an equivalence of categories $C^{\mathrm{op}} \simeq \mathbf{Fix}_C$.*

Proof. We will construct a functor $F : C^{\mathrm{op}} \to \mathbf{Fix}_C$ and show that it is full, faithful, and surjective on objects.

- Send c in C to $\mathrm{Fix}\, i_c$ in $\mathbf{Fix}_C$.

- Send $f : d \to c$ in C to $(\mathrm{Fix}\, i_d)\pi^{-1} : \mathrm{Fix}\, i_c \to \mathrm{Fix}\, i_d$ in $\mathbf{Fix}_C$, where π is an automorphism such that $\pi i_d = i_c f$, guaranteed to exist by Homogeneity. This automorphism is indeed a map $\mathrm{Fix}\, i_c \to \mathrm{Fix}\, i_d$, because if g fixes i_c then

$$\pi^{-1} g \pi i_d = \pi^{-1} g i_c f = \pi^{-1} i_c f = \pi^{-1} \pi i_d = i_d$$

  so $\pi^{-1} g \pi$ fixes i_d. The choice of π does not matter: for any other $\overline{\pi}$ with $\overline{\pi} i_d = i_c f$, it holds that $\pi^{-1}\overline{\pi} i_d = \pi^{-1} i_c f = i_d$, so $\pi^{-1}\overline{\pi} \in \mathrm{Fix}\, i_d$, so $(\mathrm{Fix}\, i_d)\pi^{-1} = (\mathrm{Fix}\, i_d)\overline{\pi}^{-1}$.
- This assignment is functorial. The identity $1_c : c \to c$ is sent to a coset $(\mathrm{Fix}\, i_c)\pi$ for some $\pi i_c = i_c 1_c$. But this means π fixes i_c, so $(\mathrm{Fix}\, i_c)\pi = \mathrm{Fix}\, i_c$. Given $f : e \to d$ and $g : d \to c$, we have $F(f) = (\mathrm{Fix}\, i_e)\pi_f^{-1}$ and $F(g) = (\mathrm{Fix}\, i_d)\pi_g^{-1}$ and $F(gf) = (\mathrm{Fix}\, i_e)\pi_{fg}^{-1}$ with π_f, π_g, and π_{fg} fitting into the following commutative rectangles:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi_f"] & \mathrm{c}_\infty \arrow[r, "\pi_g"] & \mathrm{c}_\infty \\ e \arrow[u, hook, "i_e"] \arrow[r, "f"'] & d \arrow[u, hook, "i_d"] \arrow[r, "g"'] & c \arrow[u, hook, "i_c"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{c}_\infty \arrow[rr, "\pi_{fg}"] & & \mathrm{c}_\infty \\ e \arrow[u, hook, "i_c"] \arrow[r, "f"'] & d \arrow[r, "g"'] & c \arrow[u, hook, "i_e"'] \end{tikzcd}$$

  We need $F(gf) = (\mathrm{Fix}\, i_e)\pi_{fg}^{-1} = (\mathrm{Fix}\, i_e)\pi_f^{-1}\pi_g^{-1} = F(f)F(g)$. Since in general two cosets gH, hH are equal iff $gh^{-1} \in H$, this amounts to showing $(\pi_f^{-1}\pi_g^{-1})(\pi_{fg}^{-1})^{-1} \in \mathrm{Fix}\, i_e$. The commutativity of the above rectangles implies

$$(\pi_f^{-1}\pi_g^{-1})(\pi_{fg}^{-1})^{-1} i_e = \pi_f^{-1}\pi_g^{-1}\pi_{fg} i_e = \pi_f^{-1}\pi_g^{-1} i_c g f = \pi_f^{-1}\pi_g^{-1}\pi_g \pi_f i_e = i_e$$

  so $(\pi_f^{-1}\pi_g^{-1})(\pi_{fg}^{-1})^{-1}$ fixes i_e as required.
- Full: let $(\mathrm{Fix}\, i_d)\pi^{-1}$ be a morphism $\mathrm{Fix}\, i_c \to \mathrm{Fix}\, i_d$ in $\mathbf{Fix}_C$, the goal being to find $f : d \to c$ in C such that the following square commutes:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi"] & \mathrm{c}_\infty \\ d \arrow[u, hook, "i_d"] \arrow[r, dashed, "f"'] & c \arrow[u, hook, "i_c"'] \end{tikzcd}$$

  Such an f exists iff the mono πi_d factors through i_c. By Correspondence we just need to show $\mathrm{Fix}\, i_c \subseteq \mathrm{Fix}\, \pi i_d$. This follows from the fact that $(\mathrm{Fix}\, i_d)\pi^{-1}$ is a morphism in $\mathbf{Fix}_C$: if σ fixes i_c then $\pi^{-1}\sigma\pi$ fixes i_d, so $\pi^{-1}\sigma\pi i_d = i_d$, so $\sigma\pi i_d = \pi i_d$, so σ fixes πi_d. Since σ was arbitrary we have $\mathrm{Fix}\, i_c \subseteq \mathrm{Fix}\,(\pi i_d)$ as required.
- Faithful: suppose $f, g : d \to c$ and $F(f) = F(g) : \mathrm{Fix}\, i_c \to \mathrm{Fix}\, i_d$. By definition $F(f) = (\mathrm{Fix}\, i_d)\pi_f^{-1}$ and $F(g) = (\mathrm{Fix}\, i_d)\pi_g^{-1}$ for some π_f, π_g fitting into the following commutative squares:

$$\begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi_f"] & \mathrm{c}_\infty \\ d \arrow[u, hook, "i_d"] \arrow[r, "f"'] & c \arrow[u, hook, "i_c"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{c}_\infty \arrow[r, "\pi_g"] & \mathrm{c}_\infty \\ d \arrow[u, hook, "i_d"] \arrow[r, "g"'] & c \arrow[u, hook, "i_c"] \end{tikzcd}$$

  The assumption $F(f) = F(g)$ implies $\pi_f^{-1}(\pi_g^{-1})^{-1} \in \mathrm{Fix}\, i_d$, iff $\pi_f^{-1}\pi_g \in \mathrm{Fix}\, i_d$, iff $\pi_f^{-1}\pi_g i_d = i_d$, iff $\pi_g i_d = \pi_f i_d$. Thus the bottom-left-to-top-right routes of the two squares above are equal. Commutativity of these squares implies $i_c f = i_c g$, so $f = g$ because i_c mono.
- Surjective on objects: every object $\mathrm{Fix}\, i_c$ of $\mathbf{Fix}_C$ is equal to $F(c)$, so in the image of F.

□

Lemma C.32. Let $(C, C_\infty, \mathrm{c}_\infty, i_\bullet, G)$ be a nominal situation. The set $\mathcal{U} := \{\mathrm{Fix}\, i_c \mid c \in \mathrm{Ob}(C)\}$ is cofinal in the open subgroups of the topological group G, in the sense that any open subgroup H of G contains $\mathrm{Fix}\, i_c$ for some c.

Proof. Every open subgroup H contains the identity 1_{c_∞}, so contains an open neighborhood around 1_{c_∞} of the form $\bigcap_j \mathcal{R}_{f_j}$ for $(f_j : c_j \to d_j)_{j \in J}$ a finite set of isos in C. Unwinding the definition of $\mathcal{R}$ in $1_{\mathrm{c}_\infty} \in \mathcal{R}_{f_j}$ shows $i_{d_j} f_j = i_{c_j}$ for all $j \in J$. Thus $\pi \in \mathcal{R}_{f_j}$ iff $\pi i_{c_j} = i_{d_j} f_j = i_{c_j}$ iff $\pi \in \mathrm{Fix}\, i_{c_j}$ for all $\pi \in \mathrm{Aut}(\mathrm{c}_\infty)$ and $j \in J$, so $\mathcal{R}_{f_j} = \mathrm{Fix}\, i_{c_j}$ for all $j \in J$, and by Cofinality there exists c^* with

$$H \supseteq \bigcap_j \mathcal{R}_{f_j} = \bigcap_j \mathrm{Fix}\, i_{c_j} \supseteq \mathrm{Fix}\, i_{c^*} \ni \mathcal{U}$$

as needed. □

Theorem C.33. For any nominal situation $(C, C_\infty, c_\infty, i_\bullet, G)$, there is an equivalence of categories

$$\mathrm{Sh}_{\mathrm{atomic}}(C^{\mathrm{op}}) \simeq G\,\mathbf{Set}$$

where G is given the refinement topology. Across this equivalence, an atomic sheaf F on C^{op} corresponds to an G-set $\tilde{F}$ whose carrier is $\mathrm{colim}_{c \in P^{\mathrm{op}}} Fc$ where P is the preorder of subobjects of c_∞ of the form i_c for $c \in C$. Elements of $\tilde{F}$ are equivalence classes $[c, x : Fc]$ where $[c, x] = [c', x']$ iff there exists c^* with $c \sqsubseteq c^* \sqsupseteq c'$ where $\sqsubseteq$ is the preorder P such that $x \cdot j = x' \cdot j'$, where j is the unique C-morphism witnessing $c \sqsubseteq c^*$ via the equation $i_{c^*} j = i_c$ and j' is the unique C-morphism witnessing $c' \sqsubseteq c^*$ via the equation $i_{c^*} j' = i_{c'}$. The action of G on these equivalence classes is given by $[c, x : Fc] \cdot \pi = [c', F(f)(x) : Fc']$ where $(c', f : c \to c')$ is an arbitrary iso π refines, guaranteed to exist by Closure.

Proof. $\mathbf{Fix}_C \simeq C^{\mathrm{op}}$ by Lemma C.31 and the subgroups $\mathrm{Fix}\, i_c$ are cofinal in the open subgroups of G (Lemma C.32), so MacLane and Moerdijk [30, Theorem III.9.2] gives the desired equivalence. Rifling through the proof of MacLane and Moerdijk [30, Theorem III.9.2], we find that it constructs for each $F \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{Fix}_C)$ the G-set with carrier $\mathrm{colim}_{\mathrm{Fix}\, i_c \in \mathcal{U}} F(\mathrm{Fix}\, i_c)$ where the colimit is over $\mathcal{U}$ ordered by subgroup inclusion, and the diagram the colimit is taken over sends every such inclusion $\mathrm{Fix}\, i_c \subseteq \mathrm{Fix}\, i_d$ to the $\mathbf{Fix}_C$-morphism $\mathrm{Fix}\, i_c \xrightarrow{(\mathrm{Fix}\, i_d) 1_{c_\infty}} \mathrm{Fix}\, i_d$ (MacLane and Moerdijk [30, p. 153]). By Correspondence every such morphism corresponds to the canonical morphism $i_d \to i_c$ in C_∞ / c_∞ witnessing the ordering relation $d \subseteq c$, so transporting $\mathrm{colim}_{\mathrm{Fix}\, i_c \in \mathcal{U}} F(\mathrm{Fix}\, i_c)$ across the equivalence $\mathbf{Fix}_C \simeq C^{\mathrm{op}}$ gives the colimit in the statement. Further rifling through the proof of MacLane and Moerdijk [30, Theorem III.9.2] reveals that the action of a c_∞-auto π sends an equivalence class $[\mathrm{Fix}\, i_c, x : F(\mathrm{Fix}\, i_c)]$ to $[\pi^{-1}(\mathrm{Fix}\, i_c)\pi, F(\overline{\pi})(x) : F(\pi^{-1}(\mathrm{Fix}\, i_c)\pi)]$ where $\overline{\pi}$ is the $\mathbf{Fix}_C$-morphism $\pi^{-1}(\mathrm{Fix}\, i_c)\pi \xrightarrow{(\mathrm{Fix}\, i_c)\pi} \mathrm{Fix}\, i_c$ (MacLane and Moerdijk [30, p. 153]). Pick an arbitrary c' and iso $f : c \to c'$ that π^{-1} refines, guaranteed to exist by Closure. No matter the choice of c', f, we have $\pi^{-1}(\mathrm{Fix}\, i_c)\pi = \mathrm{Fix}\, i_{c'}$, since for all σ fixing i_c it holds that $\pi^{-1}\sigma\pi i_{c'} = \pi^{-1}\sigma i_c f^{-1} = \pi^{-1} i_c f^{-1} = i_{c'} f f^{-1} = i_{c'}$ so $\pi^{-1}\sigma\pi$ fixes $i_{c'}$, and conversely for all σ fixing $i_{c'}$ it holds that $\pi\sigma\pi^{-1} i_c = \pi\sigma i_{c'} f = \pi i_{c'} f = i_c f^{-1} f = i_c$ so $\pi\sigma\pi^{-1}$ fixes i_c. Thus the $\mathbf{Fix}_C$-morphism $\overline{\pi}$ has domain $\mathrm{Fix}\, i_{c'}$ and codomain $\mathrm{Fix}\, i_c$, and the action of a c_∞-auto π sends $[\mathrm{Fix}\, i_c, x : F(\mathrm{Fix}\, i_c)]$ to $[\mathrm{Fix}\, i_{c'}, F(\overline{\pi})(x) : F(\mathrm{Fix}\, i_{c'})]$ for $(c', f : c' \to c)$ refining π^{-1}. Transporting this across the equivalence $\mathbf{Fix}_C \simeq C^{\mathrm{op}}$ gives the action in the statement: the subgroups $\mathrm{Fix}\, i_c, \mathrm{Fix}\, i_{c'}$ correspond to the objects c, c', and the morphism $\overline{\pi}$ given by the coset $(\mathrm{Fix}\, i_c)\pi$ is the image of f under the functor $C^{\mathrm{op}} \to \mathbf{Fix}_C$ constructed in the proof of Lemma C.31. ◻

D ENHANCED MEASURABLE SHEAVES

D.1 Probabilistic concepts as sheaves

Definition D.1. An *enhanced measurable sheaf* is an object of the category $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}})$.

Lemma D.2. If $(X, \mathcal{F}, \mathcal{N})$ is a standard enhanced measurable space and $(Y, \mathcal{G})$ a measurable space arising from a Polish space and $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ a measurable map, then the set $\mathcal{M} := \{G \in \mathcal{G} \mid f^{-1}(G) \in \mathcal{N}\}$ makes $(Y, \mathcal{G}, \mathcal{M})$ a standard enhanced measurable space and f a morphism in $\mathbf{EMS}_{\mathrm{std}}$.

Proof. The set $\mathcal{M}$ is a σ-ideal in $\mathcal{G}$ because $\mathcal{N}$ is a σ-ideal in $\mathcal{F}$ and taking preimages preserves all σ-algebra operations, so $(Y, \mathcal{G}, \mathcal{M})$ is an enhanced measurable space. The map f preserves and reflects negligibles by construction, since $M \in \mathcal{M}$ iff $f^{-1}(M) \in \mathcal{N}$. All that's left is to show $(Y, \mathcal{G}, \mathcal{M})$ is standard. This follows from Y Polish, by unforgetting a standard probability measure on $(X, \mathcal{F}, \mathcal{N})$ and pushing it forward onto Y through f [42, Section 2.7, p.24]. ◻

Lemma D.3. For any measurable space $(A, \mathcal{G})$ arising from a Polish space, the random variable presheaf

$$\begin{aligned}&\mathrm{RV}_A(\Omega, \mathcal{F}, \mathcal{N}) = \{\text{measurable maps } (\Omega, \mathcal{F}) \to (A, \mathcal{G})\}/{=_{\text{a.s.}}} \text{ where } X =_{\text{a.s.}} Y \text{ iff } \{\omega \mid X\omega \neq Y\omega\} \in \mathcal{N}\\ &\mathrm{RV}_A(p : \Omega' \to \Omega)([X] : \mathrm{RV}_A(\Omega)) : \mathrm{RV}_A(\Omega') = [X \circ p]\end{aligned}$$

is an atomic sheaf.

Proof. Given two random variables X, Y let $\mathrm{D}_{X,Y} := \{\omega \mid X\omega \neq Y\omega\}$ be the event they disagree. The action of RV_A on morphisms is well-defined because p negligible-reflecting: if $\mathrm{D}_{X,Y}$ negligible in Ω then $\mathrm{D}_{X \circ p, Y \circ p} = p^{-1}(\mathrm{D}_{X,Y})$ negligible in Ω'. Unwinding the definition of RV_A and making the quotienting of random variables up to almost-everywhere explicit, the presheaf RV_A is an atomic sheaf if and only if

$$\forall p\ [X].\ \underbrace{(\forall q\, r.\ p \circ q = p \circ r \implies X \circ q =_{\text{a.s.}} X \circ r)}_{X \text{ is } p\text{-invariant}} \implies \exists! [\overline{X}].\ \overline{X} p =_{\text{a.s.}} X$$

The following diagram gives the types of all variables involved:

$$\begin{tikzcd} \Omega'' \arrow[r, shift left, "q"] \arrow[r, shift right, "r"'] & \Omega' \arrow[r, "p"] \arrow[d, "X"] & \Omega \arrow[dl, "\overline{X}"] \\ & A & \end{tikzcd}$$

Suppose $[X]$ is p-invariant. By Lemma D.2, pushing the negligibles of Ω' forward along X makes A into an standard enhanced measurable space and X a morphism of standard enhanced measurable spaces. By Lemma B.26, this gives a diagram

$$\begin{tikzcd} \mathrm{alg}(\Omega') & \mathrm{alg}(\Omega) \arrow[l, hook', "p^{-1}"] \\ \mathrm{alg}(A) \arrow[u, hook, "X^{-1}"] & \end{tikzcd}$$

in $\mathbf{MbleAlg}_{\mathrm{std}}$. There is an inclusion of standard measurable algebras $\mathrm{im}(X^{-1}) \subseteq \mathrm{im}(p^{-1})$: otherwise, if there were some E in $\mathrm{im}(X^{-1})$ not in $\mathrm{im}(p^{-1})$, then by Lemma B.54 and Lemma B.26 there would exist $\mathbf{EMS}_{\mathrm{std}}$-maps $q, r : \Omega'' \to \Omega'$ with $pq = pr$ but $q^{-1}(E) \neq r^{-1}(E)$, so $X \circ q \neq_{\mathrm{a.s.}} X \circ r$, contradicting the assumption that X is p-invariant. The inclusion $\mathrm{im}(X^{-1}) \subseteq \mathrm{im}(p^{-1})$ gives a corresponding (injective) homomorphism $\mathrm{alg}(A) \hookrightarrow \mathrm{alg}(\Omega)$ making the above diagram of homomorphisms into a commutative triangle. By Lemma B.26 again, such a homomorphism arises from a measurable map $\overline{X} : \Omega \to A$, and commutativity of the triangle implies $\overline{X}p =_{\mathrm{a.s.}} X$. Finally, for any $\overline{Y} : \Omega \to A$ with $\overline{Y}p =_{\mathrm{a.s.}} X$, the string of equations $\overline{X}p =_{\mathrm{a.s.}} X =_{\mathrm{a.s.}} \overline{Y}p$ implies $\mathrm{D}_{\overline{X}p,\overline{Y}p}$ negligible in Ω'; since $\mathrm{D}_{\overline{X}p,\overline{Y}p} = p^{-1}(\mathrm{D}_{\overline{X},\overline{Y}})$ and p negligible preserving, this implies $\mathrm{D}_{\overline{Y},\overline{X}}$ negligible in Ω, which is to say $\overline{Y} =_{\mathrm{a.s.}} \overline{X}$, establishing uniqueness of $\overline{X}$. ◻

Lemma D.4. Let $U : C \to \mathbf{EMS}_{\mathrm{std}}$ be a functor and G a groupoid in C. The presheaf $よ(UG)$, more explicitly given by

$$\mathbf{EMS}_{\mathrm{std}}(\Omega, UG) = \begin{pmatrix} \{\text{maps } \Omega \to UA \text{ for some } A \in G\}/\sim \\ \text{where } (f : \Omega \to UA) \sim (g : \Omega \to UB) \text{ iff } f = U(\pi)g \text{ for some } \pi : B \to A \text{ in } G \end{pmatrix}$$

$$[f] \cdot p = [fp]$$

is an atomic sheaf on $\mathbf{EMS}_{\mathrm{std}}$.

Proof. The proof is analogous to the proof of Lemma D.3. Suppose $p : \Omega' \to \Omega$ and $[f] : \mathbf{EMS}_{\mathrm{std}}(\Omega', UG)$ is p-invariant for some $A \in G$ and $f : \Omega' \to UA$. The goal is to find $\overline{f} : \mathbf{EMS}_{\mathrm{std}}(\Omega', UG)$ with $\overline{f}p \sim f$, which is to equivalent to finding A' and $\overline{f} : \Omega' \to UA'$ and $\pi : A \to A'$ in G with $\overline{f}p = U(\pi)f$, which is equivalent (by left-multiplying both sides by $U(\pi^{-1})$) to finding $\overline{f} : \Omega' \to UA$ with $\overline{f}p = f$. As in the proof of Lemma D.3, such an $\overline{f}$ exists if there is an inclusion of measurable algebras $\mathrm{im}(f^{-1}) \subseteq \mathrm{im}(p^{-1})$, so it suffices to establish this inclusion; uniqueness of $\overline{f}$ then follows from p epi. Rephrasing the assumption that f is p-invariant in terms of measurable algebras, we have

$$\forall q\ r.\ qp^{-1} = rp^{-1} \implies \exists \pi \in G.\ qf^{-1} = rf^{-1}U(\pi)^{-1}$$

with types of the variables involved depicted by the following diagram:

$$\begin{tikzcd} \mathrm{alg}(\Omega'') & \mathrm{alg}(\Omega') \arrow[l, shift right, hook', "q"'] \arrow[l, shift left, hook', "r"] & \mathrm{alg}(\Omega) \arrow[l, hook', "p^{-1}"] \\ & \mathrm{alg}(A) \arrow[u, hook, "f^{-1}"] & \end{tikzcd}$$

Note that if $qf^{-1} = rf^{-1}U(\pi)^{-1}$ for some π then qf^{-1} and rf^{-1} have the same image in $\mathrm{alg}(\Omega'')$, so p-invariance of f implies

$$\forall q\ r.\ qp^{-1} = rp^{-1} \implies \mathrm{im}(qf^{-1}) = \mathrm{im}(rf^{-1}). \tag{3}$$

We are now ready to establish the inclusion $\mathrm{im}(f^{-1}) \subseteq \mathrm{im}(p^{-1})$. Suppose for contradiction that there exists E in $\mathrm{alg}(A)$ with $f^{-1}(E)$ not in $\mathrm{im}(p^{-1})$. By Lemma B.54, there exists a standard measurable algebra Ω'' and homomorphisms q, r as depicted above such that $qp^{-1} = rp^{-1}$ but $qf^{-1}E \notin \mathrm{im}\, r$, contradicting (3). ◻

Lemma D.5. For any standard probability space Ω, the representable functor $\mathbf{EMS}_{\mathrm{std}}(-, \Omega)$ is an atomic sheaf with minimal supports.

Proof. Sheafhood follows from Lemma D.4 by setting $U = 1$ and G to the trivial groupoid $\{1_\Omega\}$; the minimal support property follows similarly from Lemma C.18. ◻

Definition D.6 (sheaf of probability spaces)**.** The *sheaf of probability spaces* $\mathbb{P}$ is $よ(UG)$ with U the forgetful functor $\mathbf{Prob}_{\mathrm{std}} \to \mathbf{EMS}_{\mathrm{std}}$ and G the maximal subgroupoid of $\mathbf{Prob}_{\mathrm{std}}$. Concretely, the action of $\mathbb{P}$ on objects is

$$\mathbb{P}(\Omega, \mathcal{F}, \mathcal{N}) = \begin{pmatrix} \text{pairs } ((A, \mathcal{G}, \mu), X) \text{ with } (A, \mathcal{G}, \mu) \in \mathbf{Prob}_{\mathrm{std}} \text{ and } X \text{ a } \mathbf{EMS}_{\mathrm{std}}\text{-map } (\Omega, \mathcal{F}, \mathcal{N}) \to (A, \mathcal{G}, \text{negligibles}(\mu)) \\ \text{mod } \sim, \text{ where } ((A, \mathcal{G}, \mu), X) \sim ((A', \mathcal{G}', \mu'), X') \text{ iff exists } \mathbf{Prob}_{\mathrm{std}}\text{-iso } i : (A, \mathcal{G}, \mu) \to (A', \mathcal{G}', \mu') \text{ with } X' = U(i)\,X \end{pmatrix}$$

and the action on morphisms is by precomposition:

$$\mathbb{P}(f : \Omega' \to \Omega) = \begin{pmatrix} \mathbb{P}\Omega \to \mathbb{P}\Omega' \\ [(A, \mathcal{G}, \mu), X] \mapsto [(A, \mathcal{G}, \mu), Xf] \end{pmatrix}$$

This presheaf is an atomic sheaf on $\mathbf{EMS}_{\mathrm{std}}$ by Lemma D.4, and has minimal supports by Lemma C.18.

D.2 Separation as Day convolution

Lemma D.7. The category $\mathbf{EMS}_{\mathrm{std}}$ is a category of supports.

Proof. The category $\mathbf{EMS}_{\mathrm{std}}$ is symmetric semicartesian monoidal by Lemma B.35, contains only epis by Corollary B.27, and has the atomic topology by Lemma B.31 and Fact C.3. Finally, setting $U = 1_{\mathbf{EMS}_{\mathrm{std}}}$ in Lemma D.4 shows representables mod G are atomic sheaves for all groupoids G. □

Lemma D.8. The tensor product of $\mathbb{P}$ with itself in sheaves can be computed as if in presheaves:

$$i(\mathbb{P} \otimes \mathbb{P}) \cong i\mathbb{P} \otimes i\mathbb{P}$$

Proof. $\mathbf{EMS}_{\mathrm{std}}$ is a category of supports (Lemma D.7) and $\mathbb{P}$ has minimal supports, so Lemma C.23 applies. □

Lemma D.9. The tensor product $\mathbb{P} \otimes \mathbb{P}$ is a sheaf of "independent probability spaces", with action on objects

$$(\mathbb{P} \otimes \mathbb{P})(\Omega) = \left(\begin{array}{l}\text{pairs } ([A, X] : \mathbb{P}\Omega, [B, Y] : \mathbb{P}\Omega) \text{ with } X : \Omega \to \mathrm{U}A \text{ and } Y : \Omega \to \mathrm{U}B \\ \text{that factor through a tensor product; i.e., there exist } \Omega_1, \Omega_2 \text{ and } f : \Omega \to \Omega_1 \otimes \Omega_2 \text{ and } X' : \Omega_1 \to \mathrm{U}A \text{ and } Y' : \Omega_2 \to \mathrm{U}B \\ \text{with } X = X' \ \mathrm{fst}\ f \text{ and } Y = Y' \ \mathrm{snd}\ f\end{array}\right)$$

and action on morphisms

$$(\mathbb{P} \otimes \mathbb{P})(f : \Omega' \to \Omega)[(A, X), (B, Y)] = [(A, Xf), (B, Yf)].$$

Proof. The tensor product $\mathbb{P} \otimes \mathbb{P}$ is the image of the inclusion $\mathbb{P} \otimes \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$ defined by Lemma C.24. □

Definition D.10. There is a map $\mathrm{join} : \mathbb{P} \otimes \mathbb{P} \to \mathbb{P}$ defined by

$$\mathrm{join}_\Omega = \left(\begin{array}{c}\mathbb{P}\Omega \otimes \mathbb{P}\Omega \to \mathbb{P}\Omega \\ ([A, X], [B, Y]) \mapsto [A \otimes B, (X, Y)] \text{ where } (X, Y)(\omega) = (X\omega, Y\omega)\end{array}\right)$$

where we have used the representation calculated in Lemma D.9 to define join on pairs of independent probability spaces. The assumption that $[A, X]$ and $[B, Y]$ factor through some tensor product is needed in order for the map (X, Y) to be well-formed: otherwise it may fail to be negligible-reflecting as a map of enhanced measurable spaces $\Omega \to \mathrm{U}(A \otimes B)$. (For example, if $\Omega = [0, 1]$ and $X = Y = 1_{[0,1]}$, the image of the map (X, Y) is the diagonal in the unit square, a negligible set with nonnegligible preimage.)

Note D.11. Another way to arrive at the map (X, Y) in Definition D.10 is by unforgetting the tensor product that $[A, X]$ and $[B, Y]$ factor through, giving a commutative diagram

$$\begin{tikzcd}
 & \Omega \arrow[d, "f"] \arrow[dddl, bend right, "X"'] \arrow[dddr, bend left, "Y"] & \\
 & \Omega_A \otimes \Omega_B \arrow[dl, "\mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"] & \\
\Omega_A \arrow[d, "X'"'] & & \Omega_B \arrow[d, "Y'"] \\
\mathrm{U}A & & \mathrm{U}B
\end{tikzcd}$$

and then setting (X, Y) to be the composite $(X' \otimes Y')f$. (Note $\mathrm{U}(A \otimes B) = \mathrm{U}A \otimes \mathrm{U}B$ by definition, so this typechecks.) Lemma C.24, used to prove Lemma D.9, shows this construction does not depend on the choice of $f, \Omega_A, \Omega_B, X', Y'$.

Definition D.12. There is a map $\mathrm{emp} : 1 \to \mathbb{P}$ defined by

$$\mathrm{emp}_\Omega = \left(\begin{array}{l}1 \to \mathbb{P}\Omega \\ \_ \mapsto [1, ! : \Omega \to 1]\end{array}\right).$$

Lemma D.13. The tuple $(\mathbb{P}, \mathrm{join}, \mathrm{emp})$ is a commutative monoid internal to the symmetric monoidal category $(\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}), \otimes, 1)$, which is to say that the following equations hold in the internal linearly-typed language of this category:

- (Unit) $p : \mathbb{P} \vdash \mathrm{join}(\mathrm{emp}, p) = p : \mathbb{P}$
- (Commutativity) $p : \mathbb{P}, q : \mathbb{P} \vdash \mathrm{join}(p, q) = \mathrm{join}(q, p) : \mathbb{P}$
- (Associativity) $p : \mathbb{P}, q : \mathbb{P}, r : \mathbb{P} \vdash \mathrm{join}(p, \mathrm{join}(q, r)) = \mathrm{join}(\mathrm{join}(p, q), r) : \mathbb{P}$

Proof. Each of the equations holds because the corresponding property holds of $\otimes$:

- (Unit) The goal is to show $\left(\mathbb{P} \xrightarrow{\sim} 1 \otimes \mathbb{P} \xrightarrow{\mathrm{emp} \otimes 1} \mathbb{P} \otimes \mathbb{P} \xrightarrow{\mathrm{join}} \mathbb{P}\right) = 1_{\mathbb{P}}$. At stage Ω and given an element $[A, X] : \mathbb{P}\Omega$, the left side is $\mathrm{join}_\Omega([1, !_\Omega], [A, X]) = [1 \otimes A, (!_\Omega, X)]$ and the right side is $[A, X]$. These are the same equivalence classes via the isomorphism $1 \otimes A \cong A$.
- (Commutativity) The goal is to show $\left(\mathbb{P} \otimes \mathbb{P} \xrightarrow{\mathrm{join}} \mathbb{P}\right) = \left(\mathbb{P} \otimes \mathbb{P} \xrightarrow{s} \mathbb{P} \otimes \mathbb{P} \xrightarrow{\mathrm{join}} \mathbb{P}\right)$ where s witnesses symmetry of the monoidal product $\otimes$. At stage Ω and given an element of $(\mathbb{P} \otimes \mathbb{P})(\Omega)$, which by Lemma D.9 amounts to a pair of elements $([A, X], [B, Y]) : \mathbb{P}\Omega \times \mathbb{P}\Omega$ that factor through a tensor product, the left side is $\mathrm{join}_\Omega([A, X], [B, Y]) = [A \otimes B, (X, Y)]$ and the right side is $\mathrm{join}_\Omega([B, Y], [A, X]) = [B \otimes A, (Y, X)]$. These two equivalence classes are equal via the isomorphism $A \otimes B \cong B \otimes A$.
- (Associativity) The goal is to show

$$\left(\mathbb{P} \otimes (\mathbb{P} \otimes \mathbb{P}) \xrightarrow{1 \otimes \mathrm{join}} \mathbb{P} \otimes \mathbb{P} \xrightarrow{\mathrm{join}} \mathbb{P}\right) = \left(\mathbb{P} \otimes (\mathbb{P} \otimes \mathbb{P}) \xrightarrow{\sim} (\mathbb{P} \otimes \mathbb{P}) \otimes \mathbb{P} \xrightarrow{\mathrm{join} \otimes 1} \mathbb{P} \otimes \mathbb{P} \xrightarrow{\mathrm{join}} \mathbb{P}\right).$$

  At stage Ω and given an element of $(\mathbb{P} \otimes (\mathbb{P} \otimes \mathbb{P}))(\Omega)$, which by two applications of Lemma C.24 amounts to a tuple of elements $([A, X], ([B, Y], [C, Z])) : \mathbb{P}\Omega \times (\mathbb{P}\Omega \times \mathbb{P}\Omega)$ that factor through tensor products in the proper way, the left side is

$$\mathrm{join}_\Omega([A, X], \mathrm{join}_\Omega([B, Y], [C, Z])) = [A \otimes (B \otimes C), (X, (Y, Z))],$$

  and the right side is

$$\mathrm{join}_\Omega(\mathrm{join}_\Omega([A, X], [B, Y]), [C, Z]) = [(A \otimes B) \otimes C, ((X, Y), Z)].$$

  These two equivalence classes are equal via the isomorphism $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.

□

Lemma D.14. The tuple $(\mathbb{P}, \mathrm{join}, \mathrm{emp})$ is a partial commutative monoid internal to the symmetric monoidal category $(\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}), \times, 1)$, in the sense that

- Unit: for all $p : F \to \mathbb{P}$ the map $(\mathrm{emp}!, p)$ factors through i and $\mathrm{join}(\mathrm{emp}!, p) = p$.
- Commutativity: If a map $(p, q) : F \to \mathbb{P} \times \mathbb{P}$ factors through the inclusion $i : \mathbb{P} \otimes \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$, there exists a unique map $f : F \to \mathbb{P} \otimes \mathbb{P}$ with $if = (p, q)$. Abusing notation and writing f as (p, q), relying on types to disambiguate, commutativity holds if whenever (p, q) factors through i then so does (q, p), and $\mathrm{join}(p, q) = \mathrm{join}(q, p)$.
- Associativity: for all $p, q, r : F \to \mathbb{P}$ such that (p, q) factors through i and $(\mathrm{join}(p, q), r)$ factors through i, it holds that (q, r) factors through i and $(p, \mathrm{join}(q, r))$ factors through i and $\mathrm{join}(\mathrm{join}(p, q), r) = \mathrm{join}(p, \mathrm{join}(q, r))$.

Proof. A map $(p, q) : F \to \mathbb{P} \times \mathbb{P}$ factors through i iff for any $x : F\Omega$ the pair $(p(x), q(x)) : \mathbb{P}\Omega \times \mathbb{P}\Omega$ factors through a tensor product.

- (Unit) If $p(x) = [A, X]$, the pair $([1, !], [A, X])$ always factors through the tensor product $1 \otimes \mathrm{U}A$, so $(\mathrm{emp}!, p)$ always factors through i. That emp is a unit for join follows from Lemma D.13.
- (Commutativity) If $(p(x), q(x))$ factors through a tensor product $\Omega_p \otimes \Omega_q$, then $(q(x), p(x))$ factors through the tensor product $\Omega_q \otimes \Omega_p$. Thus if (p, q) factors through i then so does (q, p). Commutativity of join follows from Lemma D.13.
- (Associativity) Fix arbitrary x and write $p(x) = [A, X]$ and $q(x) = [B, Y]$ and $r(x) = [C, Z]$. Suppose $([A, X], [B, Y])$ factors through a tensor product $\Omega_p \otimes \Omega_q$ and $(\mathrm{join}([A, X], [B, Y]), [C, Z]) = ([A \otimes B, (X, Y)], [C, Z])$ factors through a tensor product $\Omega_{pq} \otimes \Omega_r$. The situation is illustrated by the following commutative diagram:

$$\begin{tikzcd}
 & & & \Omega \arrow[dll, "f_{p,q}"'] \arrow[drr, "f_{pq,r}"] & & & \\
 & \Omega_p \otimes \Omega_q \arrow[dl, "\mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"] \arrow[dd, "X' \otimes Y'"] & & & & \Omega_{pq} \otimes \Omega_r \arrow[dl, "\mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"] & \\
\Omega_p \arrow[dd, "X'"'] & & \Omega_q \arrow[dd, "Y'"] & & \Omega_{pq} \arrow[dlll, "(X,Y)'"] & & \Omega_r \arrow[d, "Z'"] \\
 & \mathrm{U}A \otimes \mathrm{U}B \arrow[dl, "\mathrm{fst}"] \arrow[dr, "\mathrm{snd}"'] & & & & & \mathrm{U}C \\
\mathrm{U}A & & \mathrm{U}B & & & &
\end{tikzcd}$$

  The root-to-leaf paths from Ω to $\mathrm{U}A, \mathrm{U}B, \mathrm{U}C$ are the random variables X, Y, Z respectively. The two paths $\Omega \to \mathrm{U}A \otimes \mathrm{U}B$ represent the random variable (X, Y): the path through $\Omega_p \otimes \Omega_q$ is the one constructed by $\mathrm{join}([A, X], [B, Y])$, as described in Note D.11, and the path through Ω_{pq} witnesses the fact that $([A \otimes B, (X, Y)], [C, Z])$ factors through the tensor product $\Omega_{pq} \otimes \Omega_r$.

With this diagram, $([B,Y],[C,Z])$ visibly factors through $\Omega_{pq} \otimes \Omega_r$. The trickier case is to show $([A,X], \mathrm{join}([B,Y],[C,Z]))$ factors through a tensor product. Contract the pentagon in the middle of the above diagram and the root-to-leaf path from Ω to $\mathrm{U}C$ to get

$$\begin{tikzcd}
 & \Omega \arrow[d, "((X,Y)'\otimes Z')f_{pq,r}"] & \\
 & (\mathrm{U}A \otimes \mathrm{U}B) \otimes \mathrm{U}C \arrow[dl, "\mathrm{fst}\ \mathrm{fst}"'] \arrow[d, "\mathrm{snd}\ \mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"'] & \\
\mathrm{U}A & \mathrm{U}B & \mathrm{U}C
\end{tikzcd}$$

Since all we have done is to contract and delete nodes in the previous diagram, the root-to-leaf paths still denote the random variables X, Y, Z. The expression $\mathrm{join}([B,Y],[C,Z]) = [B \otimes C, (Y,Z)]$ can be constructed following Note D.11, giving

$$\begin{tikzcd}
 & \Omega \arrow[d, "((X,Y)'\otimes Z')f_{pq,r}"] & & \\
 & (\mathrm{U}A \otimes \mathrm{U}B) \otimes \mathrm{U}C \arrow[dl, "\mathrm{fst}\ \mathrm{fst}"'] \arrow[d, "\mathrm{snd}\ \mathrm{fst}"'] \arrow[dr, "\mathrm{snd}\ \mathrm{fst} \otimes \mathrm{snd}"] \arrow[drr, bend left, "\mathrm{snd}"] & & \\
\mathrm{U}A & \mathrm{U}B & \mathrm{U}B \otimes \mathrm{U}C \arrow[l] \arrow[r] & \mathrm{U}C
\end{tikzcd}$$

where the path $\Omega \to \mathrm{U}B \otimes \mathrm{U}C$ is (Y,Z). Tidying and applying the isomorphism $(\mathrm{U}A \otimes \mathrm{U}B) \otimes \mathrm{U}C \cong \mathrm{U}A \otimes (\mathrm{U}B \otimes \mathrm{U}C)$ gives

$$\begin{tikzcd}
 & \Omega \arrow[d, "a((X,Y)'\otimes Z')f_{pq,r}"] & \\
 & \mathrm{U}A \otimes (\mathrm{U}B \otimes \mathrm{U}C) \arrow[dl, "\mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"] & \\
\mathrm{U}A & & \mathrm{U}B \otimes \mathrm{U}C
\end{tikzcd}$$

Since the path $\Omega \to \mathrm{U}A$ is X and the path $\Omega \to \mathrm{U}B \otimes \mathrm{U}C$ is (Y,Z), this diagram shows $(X,(Y,Z))$ factors through a tensor product as needed. Finally, the associativity equation follows from Lemma D.13.

□

Fact D.15. Let Prop denote the subobject classifier in $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}})$, which is the constant sheaf $\mathrm{Prop}(\Omega) = \{\top, \bot\}$.

Definition D.16 (ordering on probability spaces)**.** Let $(\sqsubseteq) : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$ be the map defined by

$$[A,X] \sqsubseteq_\Omega [B,Y] \iff \text{there exists a morphism } f : B \to A \text{ such that } fY = X.$$

This respects the equivalence classes $[A,X]$ and $[B,Y]$ because if $fY = X$ and $[A,X] = [\overline{A}, \overline{X}]$ and $[B,Y] = [\overline{B}, \overline{Y}]$, so $\overline{X} = iX$ and $\overline{Y} = jY$ for isos $i : A \to \overline{A}$ and $j : B \to \overline{B}$, then setting $\overline{f}$ to ifj^{-1} gives $ifj^{-1}\overline{Y} = ifj^{-1}jY = ifY = iX = \overline{X}$. And it is natural in Ω because if $p : \Omega' \to \Omega$ and $fY = X$ witnesses $[A,X] \sqsubseteq_\Omega [B,Y]$ then $fYp = Xp$ witnesses $[A,Xp] \sqsubseteq_{\Omega'} [B,Yp]$ (in words, the ordering relation is invariant under extensions of the sample space).

Theorem D.17. *The tuple $(\mathbb{P}, \mathrm{join}, \mathrm{emp}, \sqsubseteq)$ is a partially defined monoid [21, §5.3] internal to $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}})$; in other words, $(\mathbb{P}, \mathrm{join}, \mathrm{emp})$ forms an internal PCM and the following monotonicity condition holds: if $p \sqsubseteq_\Omega p'$ and $q \sqsubseteq_\Omega q'$ and (p', q') factors through a tensor product, then (p,q) does too and $\mathrm{join}_\Omega(p,q) \sqsubseteq_\Omega \mathrm{join}_\Omega(p',q')$.*

Proof. The tuple $(\mathbb{P}, \mathrm{join}, \mathrm{emp})$ is a PCM by Lemma D.14. For monotonicity, suppose $[A,X] \sqsubseteq [A',X']$ and $[B,Y] \sqsubseteq [B',Y']$ and (X',Y') factor through a tensor product. Unwinding the definition of $(\sqsubseteq)$, there exist $f : A' \to A$ and $g : B' \to B$ with $X = fX'$ and $Y = gY'$. Now consider the following commutative diagram:

$$
\begin{tikzcd}
 & \Omega \arrow[d, "p"] & \\
 & \Omega_{A'} \otimes \Omega_{B'} \arrow[dl, "\mathrm{fst}"'] \arrow[dr, "\mathrm{snd}"] \arrow[dd, "X'_* \otimes Y'_*"] & \\
\Omega_{A'} \arrow[dd, "X'_*"'] & & \Omega_{B'} \arrow[dd, "Y'_*"] \\
 & \mathrm{U}A' \otimes \mathrm{U}B' \arrow[dl] \arrow[dr] \arrow[dd, "f \otimes g"] & \\
\mathrm{U}A' \arrow[dd, "f"] & & \mathrm{U}B' \arrow[dd, "g"'] \\
 & \mathrm{U}A \otimes \mathrm{U}B \arrow[dl] \arrow[dr] & \\
\mathrm{U}A & & \mathrm{U}B
\end{tikzcd}
$$

Since (X', Y') factors through a tensor product, there exist $\Omega_{A'}, \Omega_{B'}, p, X'_*, Y'_*$ with $X' = X'_*\ \mathrm{fst}\ p$ and $Y' = Y'_*\ \mathrm{snd}\ p$ as shown. All paths from Ω to $\mathrm{U}A'$ are equal to X'. Analogously, all paths from Ω to $\mathrm{U}B'$ are equal to Y'. Since $X = fX'$ and $Y = gY'$, the root-to-leaf paths from Ω to $\mathrm{U}A$ and Ω to $\mathrm{U}B$ are X and Y respectively. The diagram shows X and Y factor through $\Omega_{A'} \otimes \Omega_{B'}$, so $\mathrm{join}_\Omega([A, X], [B, Y])$ is defined. It only remains to show $\mathrm{join}_\Omega([A, X], [B, Y]) \sqsubseteq_\Omega \mathrm{join}_\Omega([A', X'], [B', Y'])$. Since $\mathrm{join}_\Omega([A, X], [B, Y]) = [A \otimes B, (X, Y)]$ and $\mathrm{join}_\Omega([A', X'], [B', Y']) = [A' \otimes B', (X', Y')]$, it's enough to show (X, Y) factors through (X', Y'). This is visible in the diagram: the map (X, Y) is the path from Ω to $\mathrm{U}A \otimes \mathrm{U}B$, the map (X', Y') is the path from Ω to $\mathrm{U}A' \otimes \mathrm{U}B'$, and the diagram shows the factorization $(X, Y) = (f \otimes g)(X', Y')$. □

E ABSOLUTELY CONTINUOUS SETS

E.1 A nominal situation for standard enhanced measurable spaces

Definition E.1. The Hilbert cube $\mathbb{I}^\omega$ is the standard enhanced measurable space $([0, 1]^\omega, \mathcal{F}, \mathcal{N})$ of infinite sequences in the interval $[0, 1]$. The σ-algebra $\mathcal{F}$ and negligibles $\mathcal{N}$ are those obtained in constructing the usual Lebesgue measure λ on $\mathbb{I}^\omega$, given by extending the function

$$\lambda([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, 1]^\omega) = (b_1 - a_1) \times \cdots \times (b_n - a_n),$$

defined on finite-dimensional boxes in $\mathbb{I}^\omega$, to all Borel sets of $\mathbb{I}^\omega$ and then taking the completion. As with the interval $[0, 1]$, we will write $\mathbb{I}^\omega$ for both the standard enhanced measurable space and its associated standard measurable algebra, relying on context to disambiguate.

Informally speaking, our equivalence result validates the idea that a Hilbert cube's worth of randomness is enough so long as one embeds every measurable space needed into $\mathbb{I}^\omega$ in a way that leaves enough room for new randomness. To set up this result, we first introduce some auxiliary categories to help track the particular way in which a standard enhanced measurable space can be embedded into $\mathbb{I}^\omega$; these will be helpful later, when it comes to finding simple descriptions of the absolutely continuous sets corresponding to the enhanced measurable sheaves introduced in Appendix D.

Definition E.2. For $n \in \mathbb{N}$ let $\mathrm{proj}_{1..n}$ be the projection $\mathbb{I}^\omega \to [0, 1]^n$ defined by $\mathrm{proj}_{1..n}(x_1, \ldots, x_n, \ldots) = (x_1, \ldots, x_n)$.

Definition E.3. A $\mathbf{EMS}_{\mathrm{std}}$-map $f : \mathbb{I}^\omega \to Y$ has *finite footprint* if f factors through $\mathrm{proj}_{1..n}$ for some n.

Definition E.4. Let $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$ be the category whose objects are $\mathbf{EMS}_{\mathrm{std}}$-maps $\mathbb{I}^\omega \xrightarrow{p} X$ for some standard enhanced measurable space X, and whose morphisms from $\mathbb{I}^\omega \xrightarrow{p} X$ to $\mathbb{I}^\omega \xrightarrow{q} Y$ are $\mathbf{EMS}_{\mathrm{std}}$-maps $f : X \to Y$.

Definition E.5. Let $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ be the full subcategory of $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$ spanned by objects $\mathbb{I}^\omega \xrightarrow{p} X$ with finite footprint.

Note no commutativity conditions are imposed on $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$-morphisms or $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$-morphisms f in relation to their domains p and codomains q. This may make p, q seem superfluous — Lemma E.6 below establishes $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}} \simeq \mathbf{EMS}_{\mathrm{std}}$ — but having them around allows associating to each space X a particular way in which it sits inside of $\mathbb{I}^\omega$; this makes calculations easier later on.

Lemma E.6. There are equivalences $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}} \simeq \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$.

Proof. Fix an arbitrary standard enhanced measurable space X. Since X arises from a standard probability space, it is isomorphic to a coproduct Y of countably many atoms and an interval $[0, p]$. For every such coproduct Y there is at least one map $f : [0, 1] \to Y$ given by assigning a half-open subinterval to each atom and the remainder to $[0, p]$, so the composite $\left(\mathbb{I}^\omega \xrightarrow{\mathrm{proj}_{1..1}} [0,1] \xrightarrow{f} Y \xrightarrow{\sim} X\right)$ is a $\mathbf{EMS}_{\mathrm{std}}$-map $\mathbb{I}^\omega \to X$ with finite footprint. Thus for every standard enhanced measurable space X, there is at least one finite-footprint $\mathbf{EMS}_{\mathrm{std}}$-map of the form $\mathbb{I}^\omega \to X$. This establishes surjectivity-on-objects of the forgetful functor $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}} \to \mathbf{EMS}_{\mathrm{std}}$ sending $\mathbb{I}^\omega \xrightarrow{p} X$ to X; it follows that this functor witnesses $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$. The equivalence $\mathbf{EMS}_{\mathrm{std}} \simeq \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ holds similarly. ◻

Lemma E.7. For all n, there is an isomorphism of standard enhanced measurable spaces $[0,1]^n \cong [0,1]$, and hence also of their associated standard measurable algebras.

Proof. Both $[0,1]^n$ and $[0,1]$ arise from forgetting the measure on an atomless standard probability space. ◻

Lemma E.8. For all $n < m$ it holds that $[0,1]^m$ is relatively atomless over $\mathrm{im}(\mathrm{alg}(p))$, where $p : [0,1]^m \to [0,1]^n$ is the projection $p(x_1, \dots, x_n, \dots) = (x_1, \dots, x_n)$.

Proof. Every subalgebra $\mathrm{im}(\mathrm{alg}(p))$ has enough room: one can always allocate independent standard probability algebras in dimensions $n+1$ and above, so the claim follows from Theorem B.48. ◻

Definition E.9. For $n \in \mathbb{N}$ and a $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi : [0,1]^n \xrightarrow{\sim} [0,1]^n$, let $\pi \otimes 1_{\mathbb{I}^\omega} : \mathbb{I}^\omega \to \mathbb{I}^\omega$ be the automorphism defined by $(\pi \otimes 1_{\mathbb{I}^\omega})(x_0, \dots, x_{n-1}, x_n, \dots) = (y_0, \dots, y_{n-1}, x_n, \dots)$ where $\pi(x_0, \dots, x_{n-1}) = (y_0, \dots, y_{n-1})$.

Definition E.10. A $\mathbf{EMS}_{\mathrm{std}}$-automorphism $\pi : \mathbb{I}^\omega \to \mathbb{I}^\omega$ has *width n* if $\pi = \pi' \otimes 1_{\mathbb{I}^\omega}$ for some automorphism $\pi' : [0,1]^n \to [0,1]^n$.

Definition E.11. Let $\mathbf{G}^{\ll}$ be the subgroup of $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}} \mathbb{I}^\omega$ consisting of the finite-width automorphisms of $\mathbb{I}^\omega$, topologized via the refinement topology with respect to $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$-isos: for all $\mathbf{EMS}_{\mathrm{std}}$-maps $\mathbb{I}^\omega \xrightarrow{p} X$ and $\mathbb{I}^\omega \xrightarrow{q} Y$ with finite footprint and $\mathbf{EMS}_{\mathrm{std}}$-isos $f : p \xrightarrow{\sim} q$, the set $\mathcal{R}_f := \{\pi : \mathbf{G}^{\ll} \mid fp = q\pi\}$ is a basic open.

Lemma E.12. Let inc be the $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$-indexed family $\mathrm{inc}(\mathbb{I}^\omega \xrightarrow{p} X) := p$, sending each object $\mathbb{I}^\omega \xrightarrow{p} X$ of $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ to the $\mathbf{EMS}_{\mathrm{std}}$-map p, now considered as a morphism $1_{\mathbb{I}^\omega} \to p$ in $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$. The tuple $({\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}, {\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}}^{\mathrm{op}}, 1_{\mathbb{I}^\omega}, \mathrm{inc}, \mathbf{G}^{\ll})$ is a nominal situation.

Proof.

- (${\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}$ is a full subcategory of ${\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}}^{\mathrm{op}}$) By Corollary B.27.
- (${\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}$ and ${\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}}^{\mathrm{op}}$ consist only of monic maps) Both $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ and $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$ contain only epis by transporting Corollary B.27 across Lemma E.6.
- (The atomic topology exists for ${\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{opop}}$) By Lemma B.31 and Lemma E.6.
- (Closure) Fix a map $\mathrm{inc}(\mathbb{I}^\omega \xrightarrow{p} X)$, which amounts to a finite-footprint map $\mathbb{I}^\omega \xrightarrow{p} X$ factoring through $\mathrm{proj}_{1..n}$ for some n, and an auto π in $\mathbf{G}^{\ll}$, which amounts to a $\mathbf{EMS}_{\mathrm{std}}$-auto with width m. Then p also factors through $\mathrm{proj}_{1..\max(m,n)}$ and π also has width $\max(m,n)$, so $p = p'\mathrm{proj}_{1..\max(m,n)}$ and $\pi = \pi' \otimes 1_{\mathbb{I}^\omega}$ for some $p' : [0,1]^{\max(m,n)} \to X$ and $\pi' : [0,1]^{\max(m,n)} \xrightarrow{\sim} [0,1]^{\max(m,n)}$. Thus $p\pi = p'\mathrm{proj}_{1..\max(m,n)}(\pi' \otimes 1_{\mathbb{I}^\omega}) = p'\pi'\mathrm{proj}_{1..\max(m,n)}$ showing $\mathbb{I}^\omega \xrightarrow{p\pi} X$ is an object of $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$. The map $X \xrightarrow{1} X$ is a morphism $p \to p\pi$ in $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$, and the relevant square needed for Closure commutes in $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$: $\mathrm{inc}(p)\pi = p\pi = 1_X\mathrm{inc}(p\pi)$.
- (Homogeneity) Since ${\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}} \overset{\text{Lemma E.6}}{\simeq} \mathbf{EMS}^{\mathrm{op}}_{\mathrm{std}} \overset{\text{Lemma B.26}}{\simeq} \mathbf{MbleAlg}_{\mathrm{std}}$, it suffices to show that for $\mathbf{EMS}_{\mathrm{std}}$-maps $\mathbb{I}^\omega \xrightarrow{p} X$ and $\mathbb{I}^\omega \xrightarrow{q} Y$ with finite footprint and $\mathbf{MbleAlg}_{\mathrm{std}}$-morphisms $f : \mathrm{alg}(X) \to \mathrm{alg}(Y)$, there exists a finite-width automorphism $\pi : \mathbb{I}^\omega \to \mathbb{I}^\omega$ such that the following square commutes in $\mathbf{MbleAlg}_{\mathrm{std}}$:

$$\begin{tikzcd} \mathbb{I}^\omega \arrow[r, dashed, "\mathrm{alg}(\pi)"] & \mathbb{I}^\omega \\ \mathrm{alg}(X) \arrow[u, hook, "\mathrm{alg}(p)"] \arrow[r, hook, "f"'] & \mathrm{alg}(Y) \arrow[u, hook, "\mathrm{alg}(q)"'] \end{tikzcd}$$

Suppose p factors through $\mathrm{proj}_{1..n}$ and q factors through $\mathrm{proj}_{1..m}$. Then p and q also factor through $\mathrm{proj}_{1..\max(m,n)+1}$, so $p = p'\mathrm{proj}_{1..\max(m,n)+1}$ and $q = q'\mathrm{proj}_{1..\max(m,n)+1}$ for some $p' : [0,1]^{\max(m,n)+1} \to X$ and $q' : [0,1]^{\max(m,n)+1} \to Y$. It suffices to find an automorphism $\pi' : [0,1]^{\max(m,n)+1} \to [0,1]^{\max(m,n)+1}$ making the following square commute, as then $\pi' \otimes 1_{\mathbb{I}^\omega} : \mathbb{I}^\omega \to \mathbb{I}^\omega$ will be a finite-width automorphism of the form required:

$$\begin{tikzcd} {[0,1]^{\max(m,n)+1}} \arrow[r, dashed, "\mathrm{alg}(\pi')"] & {[0,1]^{\max(m,n)+1}} \\ \mathrm{alg}(X) \arrow[u, hook, "\mathrm{alg}(p')"] \arrow[r, hook, "f"'] & \mathrm{alg}(Y) \arrow[u, hook, "\mathrm{alg}(q')"'] \end{tikzcd}$$

By Lemma E.7, there is an isomorphism $i : [0,1]^{\max(m,n)+1} \cong [0,1]$, so it suffices to find an automorphism $[0,1] \to [0,1]$. Since p and q factor through $\mathrm{proj}_{1..n}$ and $\mathrm{proj}_{1..m}$, the maps p' and q' factor through the canonical projections

$$[0,1]^{\max(m,n)+1} \to [0,1]^n \qquad \text{and} \qquad [0,1]^{\max(m,n)+1} \to [0,1]^m,$$

so $[0,1]^{\max(m,n)+1}$ is relatively atomless over $\mathrm{im}(\mathrm{alg}(p'))$ and $\mathrm{im}(\mathrm{alg}(q'))$ by Lemma E.8. Hence $[0,1]$ is relatively atomless over $\mathrm{im}(i\,\mathrm{alg}(p'))$ and $\mathrm{im}(i\,\mathrm{alg}(q'))$. Lemma B.52 then gives an automorphism of the form required.

- (Correspondence) Fix $\mathbf{EMS}_{\mathrm{std}}$-maps $\mathbb{I}^\omega \xrightarrow{p} X$ and $\mathbb{I}^\omega \xrightarrow{q} Y$ with finite footprint such that $\mathrm{Fix}\,p \subseteq \mathrm{Fix}\,q$, so $p\pi = p$ implies $q\pi = q$ for all finite-width automorphisms π of $\mathbb{I}^\omega$. The goal is to show q factors through p. Since p and q have finite footprint, p factors through $\mathrm{proj}_{1..n}$ and q through $\mathrm{proj}_{1..m}$ for some n, m. Therefore, p and q also both factor through $\mathrm{proj}_{1..\max(m,n)+1}$, so $p = p'\mathrm{proj}_{1..\max(m,n)+1}$ and $q = q'\mathrm{proj}_{1..\max(m,n)+1}$ for some $p' : [0,1]^{\max(m,n)+1} \to X$ and $q' : [0,1]^{\max(m,n)+1} \to Y$. The inclusion $\mathrm{Fix}\,p \subseteq \mathrm{Fix}\,q$ implies every width-$(\max(m,n)+1)$ auto fixing p fixes q, so $p(\pi \otimes 1_{\mathbb{I}^\omega}) = p$ implies $q(\pi \otimes 1_{\mathbb{I}^\omega}) = q$ for all autos π of $[0,1]^{\max(m,n)+1}$. Since $p(\pi \otimes 1_{\mathbb{I}^\omega}) = p'\mathrm{proj}_{1..\max(m,n)+1}(\pi \otimes 1_{\mathbb{I}^\omega}) = p'\pi\mathrm{proj}_{1..\max(m,n)+1}$ and similarly $q(\pi \otimes 1_{\mathbb{I}^\omega}) = q'\mathrm{proj}_{1..\max(m,n)+1}(\pi \otimes 1_{\mathbb{I}^\omega}) = q'\pi\mathrm{proj}_{1..\max(m,n)+1}$, this implies $p'\pi = p'$ and $q'\pi = q'$ for all automorphisms π of $[0,1]^{\max(m,n)+1}$, or in other words that $\mathrm{Fix}_{\mathrm{Aut}\,[0,1]^{\max(m,n)+1}}\,p' \subseteq \mathrm{Fix}_{\mathrm{Aut}\,[0,1]^{\max(m,n)+1}}\,q'$. The maps p' and q' induce corresponding subalgebras $\mathrm{im}(\mathrm{alg}(p'))$ and $\mathrm{im}(\mathrm{alg}(q'))$ of $[0,1]^{\max(m,n)+1}$, and $[0,1]^{\max(m,n)+1}$ is relatively atomless over these subalgebras by Lemma E.8, so Lemma B.53 applies, giving an inclusion of subalgebras $\mathrm{im}(\mathrm{alg}(q')) \subseteq \mathrm{im}(\mathrm{alg}(p'))$. This in turn gives a $\mathbf{MbleAlg}_{\mathrm{std}}$-morphism $\mathrm{alg}(Y) \to \mathrm{alg}(X)$ factoring $\mathrm{alg}(q')$ through $\mathrm{alg}(p')$ (since $\mathrm{alg}(X) \cong \mathrm{im}(\mathrm{alg}(p'))$ and $\mathrm{alg}(Y) \cong \mathrm{im}(\mathrm{alg}(q'))$), which by Lemma B.26 gives a $\mathbf{EMS}_{\mathrm{std}}$-map $X \to Y$ factoring q' through p' and hence also q through p as required.
- (Cofinality) Given two objects $\mathbb{I}^\omega \xrightarrow{p} X$ and $\mathbb{I}^\omega \xrightarrow{q} Y$ of $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ with finite footprints witnessed by the factorizations $p = p'\mathrm{proj}_{1..n}$ and $q = q'\mathrm{proj}_{1..m}$ for some $n, m \in \mathbb{N}$, the map $\mathrm{proj}_{1..\max(m,n)} : \mathbb{I}^\omega \to [0,1]^{\max(m,n)}$ is relatively atomless by Lemma E.8 and there is an inclusion of subgroups $\mathrm{Fix}\,\mathrm{proj}_{1..\max(m,n)} \subseteq \mathrm{Fix}\,p \cap \mathrm{Fix}\,q$. This extends to finite families of objects by induction.

□

Theorem E.13. $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}) \overset{(1)}{\simeq} \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}) \overset{(2)}{\simeq} (\mathbf{G}^{\ll})^{\mathrm{op}}\,\mathbf{Set} \overset{(3)}{\simeq} \mathbf{G}^{\ll}\,\mathbf{Set}$.

Proof.

- (1): by Lemma E.6.
- (2): by Theorem C.33 and Lemma E.12, and $\mathrm{Aut}_{\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}{}^{\mathrm{op}}}\,\mathbb{I}^\omega = \mathrm{Aut}_{\mathbf{EMS}^{\mathrm{op}}_{\mathrm{std}}}\,\mathbb{I}^\omega = (\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega)^{\mathrm{op}}$.
- (3): by Lemma C.6.

□

Definition E.14. An *absolutely continuous set* is an object of the category $\mathbf{G}^{\ll}\,\mathbf{Set}$ of continuous $\mathbf{G}^{\ll}$-sets.

E.2 Probabilistic concepts as absolutely continuous sets

We now use the equivalence established in Theorem E.13 to calculate the $\mathbf{G}^{\ll}$-set counterparts to the sheaves defined in Appendix D.

E.2.1 Random variables.

Definition E.15. For A a Polish space, a random variable $X : \mathbb{I}^\omega \to A$ (equivalently, an element of $\mathrm{RV}_A\mathbb{I}^\omega$ where RV_A is the sheaf of random variables) has *finite footprint* if it factors through $\mathrm{proj}_{1..n}$ for some n. Write $\mathbb{I}^\omega \xrightarrow{\mathrm{ff}} A$ for the collection of random variables with finite footprint.

Definition E.16. For A a Polish space, the *absolutely continuous set of A-valued random variables* is the set $\overline{\mathrm{RV}}A := (\mathbb{I}^\omega \xrightarrow{\mathrm{ff}} A)$ of A-valued random variables with finite footprint, with action $X \cdot \pi = X\pi$.

We now show $\overline{\mathrm{RV}}_A$ indeed defines an absolutely continuous set, and moreover corresponds to the enhanced measurable sheaf RV_A.

Lemma E.17. Across the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}) \simeq \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ given by Theorem E.13, the sheaf $\mathrm{RV}_A \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}})$ of random variables corresponds to a sheaf $\widehat{\mathrm{RV}_A} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ of random variables that factor through maps with finite footprint:

$$\widehat{\mathrm{RV}_A}(p : \mathbb{I}^\omega \to \Omega) = \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid \text{there exists a unique } X' : \mathrm{RV}_A(\Omega) \text{ with } X =_{\mathrm{a.s.}} X'p\}$$
$$\widehat{\mathrm{RV}_A}(q : p' \to p)(X) = X'qp' \text{ where } X' \text{ is the unique such that } X =_{\mathrm{a.s.}} X'p$$

Proof. Let $\widetilde{\cdot}$ be the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}) \to \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$; inspecting the proof of Lemma E.6 shows that if F is a sheaf on $\mathbf{EMS}_{\mathrm{std}}$ then $\widetilde{F}$ is a sheaf on $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ defined by $\widetilde{F}(p : \mathbb{I}^\omega \to \Omega) = F(\Omega)$ on objects and $\widetilde{F}(q : p \to p') = F(q)$ on morphisms. In the case of RV_A, this gives the following:

$$\widetilde{\mathrm{RV}_A}(p : \mathbb{I}^\omega \to \Omega) = \mathrm{RV}_A(\Omega)$$
$$\widetilde{\mathrm{RV}_A}(q : p' \to p)(X) = Xq$$

We now show $\widetilde{\mathrm{RV}_A} \cong \widehat{\mathrm{RV}_A}$. For $p : \mathbb{I}^\omega \to \Omega$ an object of $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$, let α_p be the map $\widetilde{\mathrm{RV}_A}(p) \to \widehat{\mathrm{RV}_A}(p)$ that sends $X \in \widetilde{\mathrm{RV}_A}(\Omega)$ to $Xp \in \widehat{\mathrm{RV}_A}(\Omega)$. This is well-defined: it produces elements of type $\widehat{\mathrm{RV}_A}(p)$ because X is unique, for if there were some other $X' \in \mathrm{RV}_A(\Omega)$ with $Xp =_{\mathrm{a.s.}} X'p$, then $p^{-1}(\{\omega \mid X\omega \neq X'\omega\})$ negligible, so $\{\omega \mid X\omega \neq X'\omega\}$ negligible because p negligible-preserving, so $X =_{\mathrm{a.s.}} X'$. This automatically makes α_p bijective, since its inverse is the map that sends $X \in \widehat{\mathrm{RV}_A}(p)$ to the unique X' with $X =_{\mathrm{a.s.}} X'p$. Finally, α is natural in p: if $X : \Omega \to A$ and $q : p' \to p$ a morphism in $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$, then $\alpha_{p'}(\widetilde{\mathrm{RV}_A}(q)(X)) = \alpha_{p'}(Xq) = Xqp' = \widehat{\mathrm{RV}_A}(q)(Xp) = \widehat{\mathrm{RV}_A}(q)(\alpha_p(X))$. ◻

Lemma E.18. Across the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}) \simeq \mathbf{G}^{\ll}\,\mathbf{Set}$ given by Theorem E.13, the sheaf $\widehat{\mathrm{RV}_A} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ defined in Lemma E.17 corresponds to the $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega$-set $\left(\mathbb{I}^\omega \xrightarrow{\mathrm{ff}} A\right)$ of A-valued random variables with finite footprint, with action $X \cdot \pi = X\pi$.

Proof. Let $\widetilde{\cdot}$ be the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}) \to \mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega\,\mathbf{Set}$. The proofs involved in its construction are: Theorem C.33 to pass from sheaves to $\left(\mathbf{G}^{\ll}\right)^{\mathrm{op}}$-sets, and then Lemma C.6 to pass from left to right actions. Inspecting these reveals that if F is a sheaf on $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ then $\widetilde{F}$ is an $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega$-set with carrier $\operatorname{colim}_{(1_{\mathbb{I}^\omega} \xrightarrow{p} p) \in P^{\mathrm{op}}} F(p)$ where P^{op} is the set of $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$-morphisms out of $1_{\mathbb{I}^\omega}$ of the form $\mathrm{inc}(\mathbb{I}^\omega \xrightarrow{p} X)$, preordered by $(1_{\mathbb{I}^\omega} \xrightarrow{p} p) \preceq (1_{\mathbb{I}^\omega} \xrightarrow{q} q)$ if there exists a morphism r with $rp = q$ (with r necessarily unique because every $\mathbf{EMS}_{\mathrm{std}}$-map is epi). The following diagram illustrates:

$$\begin{tikzcd}
 & \mathbb{I}^\omega \arrow[d, "1_{\mathbb{I}^\omega}"] & \\
\mathbb{I}^\omega \arrow[d, "p"'] & \mathbb{I}^\omega \arrow[dl, "p"'] \arrow[dr, "q"] & \mathbb{I}^\omega \arrow[d, "q"] \\
X \arrow[rr, dashed, "r"] & & Y
\end{tikzcd}$$

The vertical arrows depict the objects $1_{\mathbb{I}^\omega}, p, q$ of $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$. The diagonal arrows are two objects of the preorder P^{op}: $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$-morphisms $1_{\mathbb{I}^\omega} \xrightarrow{p} p$ and $1_{\mathbb{I}^\omega} \xrightarrow{q} q$, equal to $\mathrm{inc}(p)$ and $\mathrm{inc}(q)$ by definition. The dashed arrow witnesses the inequality $p \preceq q$ in P^{op}: it is a $\mathbb{I}^\omega\mathbf{EMS}_{\mathrm{std}}$-morphism $p \xrightarrow{r} q$ such that $\left(1_{\mathbb{I}^\omega} \xrightarrow{p} p \xrightarrow{r} q\right) = \left(1_{\mathbb{I}^\omega} \xrightarrow{q} q\right)$, or equivalently a $\mathbf{EMS}_{\mathrm{std}}$-morphism r with $rp = q$.

Specializing to our case $F = \widehat{\mathrm{RV}_A}$ where $\widehat{\mathrm{RV}_A}$ is the sheaf defined in Lemma E.17, the carrier of $\widetilde{F} = \widetilde{\widehat{\mathrm{RV}_A}}$ is the colimit over a diagram whose $\mathrm{inc}(p)$th component (for some $p : \mathbb{I}^\omega \to \Omega$) is

$$\widehat{\mathrm{RV}_A}(\mathbb{I}^\omega \xrightarrow{p} \Omega) = \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ factors (uniquely) through } p\}.$$

Thus each component of the colimiting diagram is a subset of $\mathrm{RV}_A\mathbb{I}^\omega$. Since $\widehat{\mathrm{RV}_A}$ is an atomic sheaf, every morphism in the colimiting diagram is an injective **Set**-function (Definition C.4); unwinding definitions reveals that inequalities $(\mathbb{I}^\omega \xrightarrow{p} \Omega') \preceq (\mathbb{I}^\omega \xrightarrow{q} \Omega)$ in P^{op}, which is to say maps $r : \Omega' \to \Omega$ with $rp = q$, are sent by $\widehat{\mathrm{RV}_A}$ to inclusions

$$\begin{aligned}\{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ factors through } q\} &\hookrightarrow \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ factors through } p\}\\ (X'q \text{ for some } X' \in \mathrm{RV}_A\Omega) &\mapsto X'rp\end{aligned}$$

Since $rp = q$ by assumption, these inclusion maps are the canonical ones among subsets of $\mathrm{RV}_A\mathbb{I}^\omega$. Thus the colimiting diagram defining the carrier of $\widetilde{\widehat{\mathrm{RV}_A}}$ is a diagram of canonical inclusions of subsets of $\mathrm{RV}_A\mathbb{I}^\omega$, and its colimit is the union of all such subsets:

$$\begin{aligned}\widetilde{\widehat{\mathrm{RV}_A}} &= \bigcup_{\left(\mathbb{I}^\omega \xrightarrow{p} \Omega\right) \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}} \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ factors through } p\}\\
&= \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ factors through } p \text{ for some } \left(\mathbb{I}^\omega \xrightarrow{p} \Omega\right) \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}\}\\
&\overset{(*)}{=} \{X \in \mathrm{RV}_A\mathbb{I}^\omega \mid X \text{ has finite footprint}\}\\
&= \mathbb{I}^\omega \xrightarrow{\mathrm{ff}} A\end{aligned}$$

The equation $(*)$ holds: if X factors through a map p with finite footprint then X has finite footprint, and conversely if X has finite footprint then it factors through $\mathrm{proj}_{1..n}$ for some n. This shows the $\mathbf{G}^{\ll}$-set corresponding to $\widehat{\mathrm{RV}_A}$ has carrier $\mathbb{I}^\omega \xrightarrow{\mathrm{ff}} A$ as claimed.

Further inspecting Theorem E.13, which transports sheaves on $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ to left $\mathbf{G}^{\ll}$-sets as described in Theorem C.33, shows $\widehat{\mathrm{RV}}_A$ corresponds to the left $\mathbf{G}^{\ll}$-action on equivalence classes $[p \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}, X : \widehat{\mathrm{RV}}_A(p)]$ defined by

$$\pi \cdot \left[(\mathbb{I}^\omega \xrightarrow{p} \Omega), X : \widehat{\mathrm{RV}}_A(p)\right] = \left[(\mathbb{I}^\omega \xrightarrow{q} \Omega'), \widehat{\mathrm{RV}}_A(r)(X)\right] \text{ where } (q, r : p \to q) \text{ is an arbitrary } \mathbb{I}^\omega{\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}\text{-iso } \pi \text{ refines.}$$

The following diagram illustrates:

$$\begin{tikzcd}
\mathbb{I}^\omega \arrow[rr, "\pi"] \arrow[d, "p"'] \arrow[ddr, "X"] & & \mathbb{I}^\omega \arrow[d, dashed, "q"] \arrow[ddl, dashed, "\widehat{\mathrm{RV}}_A(r)(X)"] \\
\Omega \arrow[dr, "X'"'] & & \Omega' \arrow[ll, dashed, "r"] \\
& A &
\end{tikzcd}$$

The triangle on the left depicts the equivalence class $[p, X]$: by the calculation of the carrier of $\widetilde{\widehat{\mathrm{RV}}}_A$ above, this equivalence class corresponds to a random variable X that factors through p via X' as shown. The dashed arrows depict the action of the automorphism π on X. There exists by Closure arbitrary Ω', q, r with r a $\mathbb{I}^\omega{\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}$-iso from p to q that π refines; this amounts to Ω', q, r with $r : \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}(q, p)$ iso making the square commute as shown. The result of the action is the composite $\widehat{\mathrm{RV}}_A(r)(X) = X'rq$, illustrated by the dashed arrow $\mathbb{I}^\omega \to A$. Commutativity of the diagram and π, r iso implies $\widehat{\mathrm{RV}}_A(r)(X) = X'rq = X'p\pi^{-1} = X\pi^{-1}$. Thus the left $\mathbf{G}^{\ll}$-set corresponding to the sheaf $\widehat{\mathrm{RV}}_A$ across equivalence (2) of Theorem E.13 has action $(\pi, X) \mapsto X\pi^{-1}$. This corresponds under equivalence (3) of Theorem E.13 to the right-$\mathbf{G}^{\ll}$-action $(X, \pi) \mapsto X\pi$, as claimed. □

Theorem E.19. The enhanced measurable sheaf RV_A corresponds to the absolutely continuous set $\overline{\mathrm{RV}}_A$ across the equivalence in Theorem E.13.

Proof. Combine Lemma E.17 and Lemma E.18. □

E.2.2 Probability spaces.

Definition E.20. Let $(X, \mathcal{F}, \mathcal{N})$ be an standard enhanced measurable space, $(Y, \mathcal{G}, \mu)$ a standard probability space, and $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$ a $\mathbf{EMS}_{\mathrm{std}}$-map. The *pullback of* $(Y, \mathcal{G}, \mu)$ *along* f, written $f^*(\mathcal{G}, \mu)$, is the pair $(f^*\mathcal{G}, f^*\mu)$ defined by

$$\begin{aligned}
f^*\mathcal{G} &= \{f^{-1}(G) \triangle N \mid G \in \mathcal{G}, N' \in \mathcal{N}\} \\
f^*\mu(f^{-1}(G) \triangle N) &= \mu(G) \text{ for all } G \in \mathcal{G}, N \in \mathcal{N}
\end{aligned}$$

This operation makes $(X, f^*\mathcal{G}, f^*\mu)$ a probability space with negligibles $\mathcal{N}$ and f a measure-preserving map $(X, f^*\mathcal{G}, f^*\mu) \to (Y, \mathcal{G}, \mu)$.

Proof. The set $f^*\mathcal{G}$ is a σ-algebra: it contains the empty set because $\emptyset = f^{-1}(\emptyset) \triangle \emptyset \in f^*\mathcal{G}$, it's closed under complements because $(f^{-1}(G) \triangle N)^{\mathrm{c}} = f^{-1}(G)^{\mathrm{c}} \triangle N = f^{-1}(G^{\mathrm{c}}) \triangle N \in f^*\mathcal{G}$ for all $G \in \mathcal{G}$. For closure under countable unions, fix a countable family $(f^{-1}(G_i) \triangle N_i)_{i \in \mathbb{N}}$. First

$$\begin{aligned}
& x \in \bigcup_i (f^{-1}(G_i) \triangle N_i) \setminus \bigcup_i f^{-1}(G_i) \\
& \text{iff } (\exists i. f(x) \in G_i \Leftrightarrow x \notin N_i) \land \forall i. f(x) \notin G_i \\
& \text{iff } (\exists i. x \in N_i) \land \forall i. f(x) \notin G_i \\
& \text{iff } x \in \bigcup_i N_i \cap \bigcap_i f^{-1}(G_i^{\mathrm{c}})
\end{aligned}$$

and $\bigcup_i N_i \cap \bigcap_i f^{-1}(G_i^c)$ is in $\mathcal{N}$ because $\bigcup_i N_i$ is in $\mathcal{N}$ and $(X, \mathcal{F}, \mathcal{N})$ arises from a complete probability space. Second

$$\begin{aligned}
&x \in \bigcup_i f^{-1}(G_i) \setminus \bigcup_i (f^{-1}(G_i) \triangle N_i) \\
&\text{iff } (\exists i. f(x) \in G_i) \wedge \forall i. f(x) \in G_i \Leftrightarrow x \in N_i \\
&\text{iff } (\exists i. x \in N_i) \wedge \forall i. f(x) \in G_i \Leftrightarrow x \in N_i \\
&\text{iff } x \in \bigcup_i N_i \cap \bigcap_i f^{-1}(G_i^c) \triangle N_i
\end{aligned}$$

and $\bigcup_i N_i \cap \bigcap_i f^{-1}(G_i^c) \triangle N_i)$ negligible similarly. Thus $\underbrace{\left(\bigcup_i (f^{-1}(G_i) \triangle N_i)\right)}_{A} \triangle \underbrace{\left(\bigcup_i f^{-1}(G_i)\right)}_{B} = \underbrace{(A \setminus B)}_{\in \mathcal{N}} \uplus \underbrace{(B \setminus A)}_{\in \mathcal{N}} \in \mathcal{N}$ as required.

The measure μ' is well-defined: if $f^{-1}(G) \triangle N' = f^{-1}(\overline{G}) \triangle \overline{N}'$ for $G, \overline{G} \in \mathcal{G}$ and $N', \overline{N}' \in \mathcal{N}'$ then rearranging using the algebraic properties of $\triangle$ gives $f^{-1}(G \triangle \overline{G}) = N' \triangle \overline{N}' \in \mathcal{N}'$, so $G \triangle \overline{G} \in \mathcal{N}$ because f negligible-preserving, so $\mu(G \triangle \overline{G}) = 0$ because μ has negligibles $\mathcal{N}$, so $\mu(G) = \mu(\overline{G})$.

The equation defining $f^*\mu$ makes f measure-preserving, and $f^*\mu$ has negligibles $\mathcal{N}$ because if $G \in \mathcal{G}$ and $N \in \mathcal{N}$ then $f^*\mu(f^{-1}(G) \triangle N) = 0$ iff $\mu(G) = 0$, iff $G \in \text{negligibles}(\mu)$, iff $f^{-1}(G) \in \mathcal{N}$ (by f negligible-reflecting), iff $f^{-1}(G) \triangle N \in \mathcal{N}$. ◻

Note E.21. Informally, what this operation is doing is completing the usual pullback measure on $f^{-1}(\mathcal{G})$ with respect to the the negligibles $\mathcal{N}$. For example: if f is the map $[0, 1] \to \{\top, \bot\}$ given by the indicator function $\lambda x.[x < 1/2]$ where $[0, 1]$ is given Lebesgue-negligibles and $\{\top, \bot\}$ is given the uniform probability measure, then the usual pullback σ-algebra is the 4-element σ-algebra $\mathcal{F}$ generated by the atoms $\{[0, 1/2), [1/2, 1]\}$, with each atom getting probability $1/2$. But the operation described in Definition E.20 produces not the σ-algebra $\mathcal{F}$ but rather $\mathcal{F}$ plus all Lebesgue-negligible subsets of $[0, 1]$, so that it includes not just $[0, 1/2)$ but also the closed interval $[0, 1/2]$, sets of the form $[0, 1/2) \setminus \{x\}$ for all $x \in [0, 1]$, the set $[0, 1/2] \setminus \mathbb{Q}$, and so on.

Definition E.22 (probability space on an enhanced measurable space)**.** Let $(\Omega, \mathcal{F}, \mathcal{N})$ be an enhanced measurable space. A *probability space on* $(\Omega, \mathcal{F}, \mathcal{N})$ is a pair $(\mathcal{G}, \mu)$ such that $\mathcal{N} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and μ is a probability measure with negligibles $\mathcal{N}$. Call such a pair *standardizable* if $(\Omega, \mathcal{G}, \mu)$ arises via pullback along a map $X : (\Omega, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$ with $(Y, \mathcal{G}, \mu)$ a standard probability space. In case $(\Omega, \mathcal{F}, \mathcal{N}) = \mathbb{I}^\omega$, a standardizable probability space $(\mathcal{G}, \mu)$ has *finite footprint* if it arises by pullback along a map with finite footprint.

Definition E.23. The *absolutely continuous set of probability spaces* is the set $\overline{\mathbb{P}}$ of standardizable probability spaces on $\mathbb{I}^\omega$ with finite footprint, and action $(\mathcal{F}, \mu) \cdot \pi = \pi^*(\mathcal{F}, \mu)$.

We now show $\overline{\mathbb{P}}$ indeed defines an absolutely continuous set, and moreover corresponds to the enhanced measurable sheaf $\mathbb{P}$.

Lemma E.24. Let $(\Omega, \mathcal{F}, \mathcal{N})$ be an enhanced measurable space, $X : (\Omega, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(A, \mathcal{G}, \mu)$ a $\mathbf{EMS}_{\mathrm{std}}$-map, and $\pi : (A, \mathcal{G}, \mu) \to (A', \mathcal{G}', \mu')$ a $\mathbf{Prob}_{\mathrm{std}}$-iso. Then $X^*(\mathcal{G}, \mu) = (\pi X)^*(\mathcal{G}', \mu')$.

Proof. Let $\tau : (A', \mathcal{G}', \mu') \to (A, \mathcal{G}, \mu)$ be the inverse of π (to avoid confusion with the operation π^{-1} of taking π-preimages). First, the σ-algebras $X^*\mathcal{G}$ and $(\pi X)^*\mathcal{G}'$ are equal. For any event $X^{-1}(G) \triangle N \in X^*\mathcal{G}$ with $G \in \mathcal{G}, N \in \mathcal{N}$,

$$X^{-1}(G) \triangle (\pi X)^{-1}(\tau^{-1}G) = X^{-1}(G) \triangle X^{-1}(\pi^{-1}(\tau^{-1}(G)) = X^{-1}(G) \triangle X^{-1}(M)$$

for some $M \in \text{negligibles}(\mu)$, so $X^{-1}(G) \triangle N = (\pi X)^{-1} \underbrace{(\tau^{-1}G)}_{\in \mathcal{G}'} \triangle \underbrace{(N \triangle X^{-1}(M))}_{\in \mathcal{N}} \in X^*\mathcal{G}'$. This shows $X^*\mathcal{G} \subseteq (\pi X)^*\mathcal{G}'$. Running the same argument with the roles of π and τ swapped shows the converse inclusion. Next, the measures are equal: for $X^{-1}(G) \triangle N \in X^*\mathcal{G}$ with $G \in \mathcal{G}, N \in \mathcal{N}$,

$$\begin{aligned}
(\pi X)^*\mu'(X^{-1}(G) \triangle N) &= (\pi X)^*\mu'((\pi X)^{-1} \underbrace{(\tau^{-1}G)}_{\in \mathcal{G}'} \triangle \underbrace{X^{-1}(M)}_{\in \mathcal{N}}) \text{ for some } M \in \text{negligibles}(\mu)\text{, as in the argument above} \\
&= \mu'(\tau^{-1}(G)) \text{ by definition of } (\pi X)^*\mu' \\
&= \mu(G) \text{ because } \tau \text{ measure-preserving} \\
&= X^*\mu(X^{-1}(G) \triangle N).
\end{aligned}$$

◻

Lemma E.25. Pullback of probability spaces respects respects composition of $\mathbf{EMS}_{\mathrm{std}}$-maps: for all $\mathbf{EMS}_{\mathrm{std}}$-maps $f : (X, \mathcal{F}, \mathcal{N}) \to (Y, \mathcal{G}, \mathcal{M})$ and $g : (Y, \mathcal{G}, \mathcal{M}) \to \mathrm{U}(Z, \mathcal{H}, \mu)$ it holds that $(gf)^*(\mathcal{H}, \mu) = f^*g^*(\mathcal{H}, \mu)$.

Proof. First, the σ-algebras are equal. Every $(gf)^{-1}(H)\triangle N \in (gf)^*\mathcal{H}$ is equal to $f^{-1}(g^{-1}(H)\triangle\emptyset)\triangle N \in f^*g^*\mathcal{H}$ and conversely every $f^{-1}(g^{-1}(H)\triangle M)\triangle N \in f^*g^*\mathcal{H}$ is equal to $(gf)^{-1}(H)\triangle \underbrace{f^{-1}(M)\triangle N}_{\in\mathcal{N}} \in (gf)^*\mathcal{H}$. Second, the measures are equal:

$$f^*g^*\mu(f^{-1}(g^{-1}(H)\triangle M)\triangle N) = g^*\mu(g^{-1}(H)\triangle M) = \mu(H) = (gf)^*\mu((gf)^{-1}(H)\triangle(f^{-1}(M)\triangle N)) = (gf)^*\mu(f^{-1}(g^{-1}(H)\triangle M)\triangle N)$$

for all $(f^{-1}(g^{-1}(H)\triangle M)\triangle N) \in f^*g^*\mathcal{H}$. ◻

Lemma E.26. For $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(Y, \mathcal{G}, \mu)$ a $\mathbf{EMS}_{\mathrm{std}}$-map, the measure algebra homomorphism $\mathrm{alg}(X, f^*\mathcal{G}, f^*\mu) \xleftarrow{\mathrm{alg}(f)} \mathrm{alg}(Y, \mathcal{G}, \mu)$ is bijective.

Proof. The homomorphism $\mathrm{alg}(f)$ is automatically injective because f measure-preserving as a map $(X, f^*\mathcal{G}, f^*\mu) \to (Y, \mathcal{G}, \mu)$ and measure-preserving homomorphisms are injective [19, 324K(a)]. It's surjective because every element of $\mathrm{alg}(X, f^*\mathcal{G}, f^*\mu)$ is an equivalence class of the form $[f^{-1}(G)\triangle N]$ in the quotient boolean algebra $f^*\mathcal{G}/\mathrm{negligibles}(f^*\mu)$ for some $G \in \mathcal{G}, N \in \mathcal{N}$, and

$$[f^{-1}(G)\triangle N] \stackrel{(*)}{=} [f^{-1}(G)] = \mathrm{alg}(f)[G]$$

where $(*)$ holds because $f^*\mu$ has negligibles $\mathcal{N}$. ◻

Lemma E.27. The sheaf $\mathbb{P}$ is equivalently a sheaf of standardizable probability spaces, whose action on objects is

$$\mathbb{P}\Omega \cong \{\text{standardizable probability spaces } (\mathcal{G}, \mu) \text{ on } \Omega\}$$

and whose action on morphisms takes pullbacks of probability spaces:

$$\mathbb{P}(f : \Omega' \to \Omega)(\mathcal{G}, \mu) = f^*(\mathcal{G}, \mu)$$

Proof. Any $[(A, \mathcal{F}, \mu), X : \Omega \to A] \in \mathbb{P}\Omega$ gives rise to a standardizable probability space $X^*(\mathcal{F}, \mu)$. This operation respects the equivalence class $[A, X]$ by Lemma E.24, and is surjective by definition of standardizable probability space. It is natural in Ω by Lemma E.25. All that's left is to show injectivity. Fix $[(A, \mathcal{F}, \mu), X], [(B, \mathcal{G}, \nu), Y]$ with $(X^*\mathcal{F}, X^*\mu) = (Y^*\mathcal{G}, Y^*\nu)$. Applying Lemma E.26 to X and Y gives bijective measure-algebra homomorphisms

$$\mathrm{alg}(X) : \mathrm{alg}(A, \mathcal{F}, \mu) \to \mathrm{alg}(\Omega, X^*\mathcal{F}, X^*\mu) \qquad \text{and} \qquad \mathrm{alg}(Y) : \mathrm{alg}(B, \mathcal{G}, \nu) \to \mathrm{alg}(\Omega, Y^*\mathcal{G}, Y^*\nu).$$

The composition $i^* := \mathrm{alg}(X)^{-1}\,\mathrm{alg}(Y) : \mathrm{alg}(B, \mathcal{G}, \nu) \to \mathrm{alg}(A, \mathcal{F}, \mu)$, well-typed because $(X^*\mathcal{F}, X^*\mu) = (Y^*\mathcal{G}, Y^*\nu)$, satisfies $\mathrm{alg}(X)i^* = \mathrm{alg}(Y)$, so corresponds by Lemma B.26 to a $\mathbf{Prob}_{\mathrm{std}}$-iso $i : (A, \mathcal{F}, \mu) \to (B, \mathcal{G}, \nu)$ such that $\mathrm{U}(i)X = Y$, witnessing $[(A, \mathcal{F}, \mu), X] = [(B, \mathcal{G}, \nu), Y]$. ◻

Lemma E.28. For $f : (X, \mathcal{F}, \mathcal{N}) \to (Y, \mathcal{G}, \mathcal{M})$ a $\mathbf{EMS}_{\mathrm{std}}$-map and $(\mathcal{H}, \mu), (\mathcal{H}', \mu')$ probability spaces on $(Y, \mathcal{G}, \mathcal{M})$, if $f^*(\mathcal{H}, \mu) = f^*(\mathcal{H}', \mu')$ then $(\mathcal{H}, \mu) = (\mathcal{H}', \mu')$.

Proof. If $H \in \mathcal{H}$ then $f^{-1}(H) = f^{-1}(H)\triangle\emptyset \in f^*\mathcal{H} = f^*\mathcal{H}'$, so $f^{-1}(H) = f^{-1}(H')\triangle N$ for some $H' \in \mathcal{H}'$ and $N \in \mathcal{N}$, so $f^{-1}(H)\triangle f^{-1}(H') = f^{-1}(H\triangle H') = N \in \mathcal{N}$, so $H\triangle H' \in \mathcal{M}$ since f negligible-preserving, so $H\triangle H' \in \mathcal{H}'$ since $\mathcal{H}' \supseteq \mathcal{M}$, so $H'\triangle(H\triangle H') = H \in \mathcal{H}'$. This shows $\mathcal{H} \subseteq \mathcal{H}'$; the converse inclusion follows from an analogous argument with the roles of $\mathcal{H}, \mathcal{H}'$ swapped. Thus $\mathcal{H} = \mathcal{H}'$. Finally, if $H \in \mathcal{H}$ then $f^{-1}(H) = f^{-1}(H')\triangle N$ for some $H' \in \mathcal{H}', N \in \mathcal{N}$, so $\mu(H) = f^*\mu(f^{-1}(H)) = f^*\mu'(f^{-1}(H')\triangle N) = \mu'(H') = \mu(H)$ where the last equality follows from $f^{-1}(H)\triangle f^{-1}(H') = N \in \mathcal{N}$ and f negligible-preserving and $\mathrm{negligibles}(\mu) = \mathrm{negligibles}(\mu') = \mathcal{M}$. ◻

Lemma E.29. Across the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}) \simeq \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ given by Theorem E.13, the sheaf $\mathbb{P} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}})$ of standardizable probability spaces described in Lemma E.27 corresponds to a sheaf $\widehat{\mathbb{P}} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ of standardizable probability spaces that factor through maps with finite footprint:

$$\widehat{\mathbb{P}}(p : \mathbb{I}^\omega \to \Omega) = \{(\mathcal{G}, \nu) \text{ a standardizable probability space on } \mathbb{I}^\omega \mid \text{there exists a unique } (\mathcal{F}, \mu) \text{ on } \Omega \text{ with } (\mathcal{G}, \nu) = p^*(\mathcal{F}, \mu)\}$$
$$\widehat{\mathbb{P}}(q : p' \to p)(p^*(\mathcal{F}, \mu)) = (qp')^*(\mathcal{F}, \mu)$$

Proof. The proof is similar to Lemma E.17. Let $\widetilde{\cdot}$ be the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{EMS}_{\mathrm{std}}) \to \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$; across this equivalence and Lemma E.27, the sheaf $\widetilde{\mathbb{P}} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ is

$$\widetilde{\mathbb{P}}(p : \mathbb{I}^\omega \to \Omega) = \{\text{standardizable probability spaces } (\mathcal{G}, \mu) \text{ on } \Omega\}$$
$$\widetilde{\mathbb{P}}(q : p' \to p)(\mathcal{F}, \mu) = q^*(\mathcal{F}, \mu)$$

We now show $\widetilde{\mathbb{P}} \cong \widehat{\mathbb{P}}$. For $p : \mathbb{I}^\omega \to \Omega$ an object of $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$, let α_p be the map $\widetilde{\mathbb{P}}(p) \to \widehat{\mathbb{P}}(p)$ that sends $(\mathcal{F}, \mu) \in \widetilde{\mathbb{P}}(\Omega)$ to $p^*(\mathcal{F}, \mu) \in \widehat{\mathbb{P}}(\Omega)$. This is well-defined: it produces elements of type $\widehat{\mathbb{P}}(p)$ because $(\mathcal{F}, \mu)$ is unique by Lemma E.28. This automatically makes α_p bijective, with inverse the map sending $(\mathcal{G}, \nu) \in \widehat{\mathbb{P}}(p)$ to the unique $(\mathcal{F}, \mu) \in \widetilde{\mathbb{P}}(p)$ with $(\mathcal{G}, \nu) = p^*(\mathcal{F}, \mu)$. Finally, α is natural in p: if $(\mathcal{F}, \mu) \in \widetilde{\mathbb{P}}(p)$

and $q : p' \to p$ a $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$-morphism, then $\alpha_{p'}(\widetilde{\mathbb{P}}(q)(\mathcal{F},\mu)) = \alpha_{p'}(q^*(\mathcal{F},\mu)) = (qp')^*(\mathcal{F},\mu) = \widehat{\mathbb{P}}(q)(p^*(\mathcal{F},\mu)) = \widehat{\mathbb{P}}(q)(\alpha_p(\mathcal{F},\mu))$ by Lemma E.25. □

Lemma E.30. Across the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}) \simeq \mathbf{G}^{\ll}\ \mathbf{Set}$ given by Theorem E.13, the sheaf $\widehat{\mathbb{P}} \in \mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}})$ defined in Lemma E.29 corresponds to the $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega$-set of standardizable probability spaces on $\mathbb{I}^\omega$ with finite footprint, with action $(\mathcal{F},\mu)\cdot\pi = \pi^*(\mathcal{F},\mu)$.

Proof. The proof is similar to Lemma E.18. Let $\widetilde{-}$ be the equivalence $\mathrm{Sh}_{\mathrm{atomic}}(\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}) \to \mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\,\mathbb{I}^\omega\ \mathbf{Set}$. The proofs involved in its construction are: Theorem C.33 to pass from sheaves to $(\mathbf{G}^{\ll})^{\mathrm{op}}$-sets, and then Lemma C.6 to pass from left to right actions. As in the proof of Lemma E.18, the sheaf $\widehat{\mathbb{P}}$ is sent to an $\mathbf{G}^{\ll}$-set $\widetilde{\widehat{\mathbb{P}}}$ whose carrier is a colimit over $\widehat{\mathbb{P}}(p)$ as p ranges over the preorder on $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$-objects of the form $\mathbb{I}^\omega \xrightarrow{p} p$ with ordering relation $(\mathbb{I}^\omega \xrightarrow{p} p) \preceq (\mathbb{I}^\omega \xrightarrow{q} q)$ iff $rp = q$ for some (necessarily unique) r. For $\mathbb{I}^\omega \xrightarrow{p} p$,

$$\widehat{\mathbb{P}}(\mathbb{I}^\omega \xrightarrow{p} \Omega) = \{(\mathcal{G},\nu) \text{ on } \mathbb{I}^\omega \mid \text{there exists a unique } (\mathcal{F},\mu) \text{ with } (\mathcal{G},\nu) = p^*(\mathcal{F},\mu)\},$$

so all objects in the colimiting diagram are subsets of the set of standardizable probability spaces on $\mathbb{I}^\omega$. Ordering relations $(\mathbb{I}^\omega \xrightarrow{p} \Omega') \preceq (\mathbb{I}^\omega \xrightarrow{q} \Omega)$, which is to say maps $r : \Omega' \to \Omega$ with $rp = q$, are sent by $\widehat{\mathbb{P}}$ to inclusions

$$\begin{aligned} \widehat{\mathbb{P}}(q) &\hookrightarrow \widehat{\mathbb{P}}(p) \\ q^*(\mathcal{F},\mu) &\mapsto (rp)^*(\mathcal{F},\mu) = q^*(\mathcal{F},\mu) \text{ since } rp = q \end{aligned}$$

so morphisms of the colimiting diagram are the canonical ones among subsets of the set of standardizable probability spaces on $\mathbb{I}^\omega$. Thus the colimit defining the carrier of $\widetilde{\widehat{\mathrm{RV}}_A}$ is the union of all such subsets:

$$\begin{aligned} \widetilde{\widehat{\mathbb{P}}} &= \bigcup_{(\mathbb{I}^\omega \xrightarrow{p} \Omega) \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}} \{(\mathcal{G},\nu) \text{ standardizable on } \mathbb{I}^\omega \mid (\mathcal{G},\nu) \text{ factors through } p\} \\ &= \{(\mathcal{G},\nu) \text{ standardizable on } \mathbb{I}^\omega \mid (\mathcal{G},\nu) = p^*(\mathcal{F},\mu) \text{ for some } (\mathbb{I}^\omega \xrightarrow{p} \Omega) \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}} \text{ and } (\mathcal{F},\mu) \text{ standardizable on } \Omega\} \\ &= \{(\mathcal{G},\nu) \text{ standardizable on } \mathbb{I}^\omega \mid (\mathcal{G},\nu) \text{ has finite footprint}\} \end{aligned}$$

Further inspecting Theorem E.13, which transports sheaves on $\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}$ to left $\mathbf{G}^{\ll}$-sets as described in Theorem C.33, shows $\widehat{\mathbb{P}}$ corresponds to the left $\mathbf{G}^{\ll}$-action on equivalence classes $[p \in \mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}, (\mathcal{G},\nu) : \widehat{\mathbb{P}}(p)]$ defined by

$$\pi \cdot \left[(\mathbb{I}^\omega \xrightarrow{p} \Omega), (\mathcal{G},\nu) : \widehat{\mathbb{P}}(p)\right] = \left[(\mathbb{I}^\omega \xrightarrow{q} \Omega'), \widehat{\mathbb{P}}(r)(\mathcal{G},\nu)\right] \text{ where } (q, r : p \to q) \text{ is an arbitrary } {\mathbb{I}^\omega\mathbf{EMS}^{\mathrm{ff}}_{\mathrm{std}}}^{\mathrm{op}}\text{-iso } \pi \text{ refines.}$$

In this case, this action amounts to sending a standardizable probability space $p^*(\mathcal{F},\mu)$ (where p is some map $\mathbb{I}^\omega \to \Omega$ with finite footprint) to $(rq)^*(\mathcal{F},\mu)$ where q, r arbitrary such that r iso and

$$\begin{tikzcd} \mathbb{I}^\omega \arrow[r, "\pi"] \arrow[d, "p"'] & \mathbb{I}^\omega \arrow[d, "q"] \\ \Omega & \Omega' \arrow[l, "r"] \end{tikzcd}$$

commutes. This implies $rq = p\pi^{-1}$, so the action sends $p^*(\mathcal{F},\mu)$ to $(p\pi^{-1})^*(\mathcal{F},\mu) = (\pi^{-1})^* p^*(\mathcal{F},\mu)$ (Lemma E.25). Since every element of $\widetilde{\widehat{\mathbb{P}}}$ is of the form $p^*(\mathcal{F},\mu)$ for some $p, \mathcal{F}, \mu$, this shows the left $\mathbf{G}^{\ll}$-set corresponding to the sheaf $\widehat{\mathbb{P}}$ across equivalence (2) of Theorem E.13 has action $(\pi, (\mathcal{G},\nu)) \mapsto (\pi^{-1})^*(\mathcal{G},\nu)$. This corresponds under equivalence (3) of Theorem E.13 to the right $\mathbf{G}^{\ll}$-action $((\mathcal{G},\nu),\pi) \mapsto \pi^*(\mathcal{G},\nu)$, as claimed. □

Theorem E.31. The enhanced measurable sheaf $\mathbb{P}$ corresponds to the absolutely continuous set $\overline{\mathbb{P}}$ across the equivalence in Theorem E.13.

Proof. Combine Lemma E.29 and Lemma E.30. □

E.3 Separation as independent combination

Appendix D.2 established the following:

- The Day convolution $\mathbb{P} \otimes \mathbb{P}$ is a subobject of the sheaf $\mathbb{P} \times \mathbb{P}$ of pairs of probability spaces, consisting of those pairs which factor through a tensor product.
- There is a map $\mathrm{join} : \mathbb{P} \otimes \mathbb{P} \to \mathbb{P}$ combining such pairs of probability spaces. Viewed as a partial map $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$ and in combination with an ordering relation $(\sqsubseteq) : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$, it forms a partially defined monoid (PDM) internal to the category of enhanced measurable sheaves (Theorem D.17).

By the equivalence of enhanced measurable sheaves and absolutely continuous sets in Theorem E.13, there is a corresponding PDM internal to the category of absolutely continuous sets. We show this PDM is — modulo small differences regarding negligible sets — the PDM of Li et al. [29], and in particular that the monoidal operation is independent combination.

Definition E.32 (independent sub-σ-algebras [19, 272A(b)])**.** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Two sub-σ-algebras $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ are *independent* if $\mu(G \cap H) = \mu(G)\mu(H)$ for all $G \in \mathcal{G}, H \in \mathcal{H}$.

Definition E.33 (ordering on probability spaces)**.** Let $(\Omega, \mathcal{F}, \mathcal{N})$ be a standard enhanced measurable space. Let $\overline{\sqsubseteq}$ be the ordering on standardizable probability spaces on Ω given by $(\mathcal{G}, \mu) \mathrel{\overline{\sqsubseteq}} (\mathcal{G}', \mu')$ iff $\mathcal{G} \subseteq \mathcal{G}'$ and $\mu = \mu'|_{\mathcal{G}}$.

Definition E.34 (independent combination)**.** Let $(\Omega, \mathcal{F}, \mathcal{N})$ be a standard enhanced measurable space. Two standardizable probability spaces $(\mathcal{G}, \mu), (\mathcal{H}, \nu)$ on $(\Omega, \mathcal{F}, \mathcal{N})$ (Definition E.22) are *independently combinable* if there exists a standardizable probability space $(\mathcal{K}, \rho)$ on $(\Omega, \mathcal{F}, \mathcal{N})$ such that $(\mathcal{G}, \mu) \mathrel{\overline{\sqsubseteq}} (\mathcal{K}, \rho) \mathrel{\overline{\sqsupseteq}} (\mathcal{H}, \nu)$ and $\mathcal{G}$ and $\mathcal{H}$ are independent sub-σ-algebras in the probability space $(\Omega, \mathcal{K}, \rho)$. If $(\mathcal{K}, \rho)$ is the smallest such standardizable probability space with respect to the ordering $\overline{\sqsubseteq}$, then it is called the *independent combination* of $(\mathcal{G}, \mu)$ and $(\mathcal{H}, \nu)$.

Compare Definition E.34 with Li et al. [29, Definition 2.2]: they are essentially the same, up to the σ-ideal $\mathcal{N}$ of negligible sets and the stipulation that probability spaces be standardizable.

Definition E.35 (empty probability space)**.** Let $\mathcal{N}$ be the negligibles of $\mathbb{I}^\omega$. The *empty probability space on* $\mathbb{I}^\omega$, written $\overline{\mathrm{emp}}$, is the probability space on $\mathbb{I}^\omega$ with σ-algebra generated by $\mathcal{N}$ and measure defined by $\mu(N) = 0$ for all $N \in \mathcal{N}$, standardizable because it arises via pullback along the unique $\mathbf{EMS}_{\mathrm{std}}$-map $! : (\Omega, \mathcal{F}, \mathcal{N}) \to (1, \{\emptyset, 1\}, \emptyset)$, finite-footprint because ! factors through $\mathrm{proj}_{1..0}$.

In Appendix E.2.2 we showed that, across the equivalence in Theorem E.13, the sheaf $\mathbb{P}$ corresponds to the absolutely continuous set $\overline{\mathbb{P}}$ of standardizable probability spaces with finite footprint. In this section we extend this correspondence with the following:

- The map $\mathrm{emp} : 1 \to \mathbb{P}$ corresponds to the empty probability space $\overline{\mathrm{emp}} : \overline{\mathbb{P}}$ (equivalently a map $1 \to \overline{\mathbb{P}}$ in the category of absolutely continuous sets) (Lemma E.36).
- The ordering relation $\sqsubseteq : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$ corresponds to the ordering $\overline{\sqsubseteq} : \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \mathrm{Prop}$ (Lemma E.40).
- The Day convolution $\mathbb{P} \otimes \mathbb{P}$ corresponds to an absolutely continuous set $\overline{\mathbb{P}}^2_{\perp}$ of pairs of independently-combinable probability spaces (Lemma E.44).
- The partial map $\mathrm{join} : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$ corresponds to a partial map $\overline{\mathrm{join}} : \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \mathrm{Prop}$ that sends independently-combinable pairs to their independent combination (Theorem E.47).

Putting this together shows the PDM $(\mathbb{P}, \mathrm{join}, \mathrm{emp}, \sqsubseteq)$ internal to enhanced measurable sheaves corresponds to the PDM $(\overline{\mathbb{P}}, \overline{\mathrm{join}}, \overline{\mathrm{emp}}, \overline{\sqsubseteq})$ internal to absolutely continuous sets (Theorem E.48).

The recipe for showing a morphism of enhanced measurable sheaves corresponds to a morphism of absolutely continuous sets across the equivalence in Theorem E.13 is as follows. For the sheaves F introduced in Appendix D, the corresponding absolutely continuous set $\overline{F}$ is a union of the images of the embeddings $F(p) : F\Omega \hookrightarrow F\mathbb{I}^\omega$ for $\mathbf{EMS}_{\mathrm{std}}$-maps $p : \mathbb{I}^\omega \to \Omega$ with finite footprint. A natural transformation of enhanced measurable sheaves $\alpha : F \to G$ then corresponds to a map of absolutely continuous sets $f : \overline{F} \to \overline{G}$ across the equivalence in Theorem E.13 if for all Ω and $p : \mathbb{I}^\omega \to \Omega$ with finite footprint, it holds that $\left(F\Omega \xhookrightarrow{F(p)} \overline{F} \xrightarrow{f} \overline{G}\right) = \left(F\Omega \xrightarrow{\alpha_\Omega} G\Omega \xhookrightarrow{G(p)} \overline{G}\right)$; i.e., f behaves like α on elements $x \in F\Omega$ when embedded into $\overline{F}$ via $F(p)$.

Lemma E.36. The constant function at the empty probability space $\overline{\mathrm{emp}}$ defines a map $1 \to \overline{\mathbb{P}}$ of absolutely continuous sets, and this map corresponds to the map $\mathrm{emp} : 1 \to \mathbb{P}$ across the equivalence in Theorem E.13.

Proof. At stage $\Omega \in \mathbf{EMS}_{\mathrm{std}}$, the natural transformation emp picks out the equivalence class $[1, ! : \Omega \to \mathrm{U}1] \in \mathbb{P}\Omega$ where $1 \in \mathbf{Prob}_{\mathrm{std}}$ is the one-point probability space. For any $\mathbf{EMS}_{\mathrm{std}}$-map $p : \mathbb{I}^\omega \to \Omega$ with finite footprint, the element $\mathbb{P}(p)[1, !]$ of $\overline{\mathbb{P}}$ corresponding to $[1, !]$ is the pullback $(!p)^* 1$ on $\mathbb{I}^\omega$; this is precisely $\overline{\mathrm{emp}}$. ◻

Lemma E.37. Let $(X, \mathcal{F}, \mathcal{N})$ be an enhanced measurable space. Let $f : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(A, \mathcal{G}, \mu)$ and $g : (X, \mathcal{F}, \mathcal{N}) \to \mathrm{U}(B, \mathcal{H}, \nu)$ be two $\mathbf{EMS}_{\mathrm{std}}$-maps. If $f^*(\mathcal{G}, \mu) \mathrel{\overline{\sqsubseteq}} f^*(\mathcal{H}, \nu)$ then there exists a $\mathbf{Prob}_{\mathrm{std}}$-map $p : (B, \mathcal{H}, \nu) \to (A, \mathcal{G}, \mu)$ such that $f = \mathrm{U}(p)g$.

Proof. The inequality $f^*(\mathcal{G}, \mu) \mathrel{\overline{\sqsubseteq}} f^*(\mathcal{H}, \nu)$ implies a corresponding inclusion of measure algebras $i : \mathrm{alg}(f^*(\mathcal{G}, \mu)) \hookrightarrow \mathrm{alg}(f^*(\mathcal{H}, \nu))$. By Lemma E.26, the maps f and g give rise to $\mathbf{Prob}_{\mathrm{std}}$-isos

$$\mathrm{alg}(f) : \mathrm{alg}(A, \mathcal{G}, \mu) \xrightarrow{\sim} \mathrm{alg}(f^*(\mathcal{G}, \mu)) \qquad \text{and} \qquad \mathrm{alg}(g) : \mathrm{alg}(B, \mathcal{H}, \nu) \xrightarrow{\sim} \mathrm{alg}(g^*(\mathcal{H}, \nu)).$$

The composite $p^* := \mathrm{alg}(g)^{-1} i \,\mathrm{alg}(f)$ is a $\mathbf{ProbAlg}_{\mathrm{std}}$-morphism $\mathrm{alg}(A, \mathcal{G}, \mu) \to \mathrm{alg}(B, \mathcal{H}, \nu)$ with $\mathrm{alg}(g)p^* = i \,\mathrm{alg}(f)$, so by Lemma B.26 corresponds to a $\mathbf{Prob}_{\mathrm{std}}$-map $p : (B, \mathcal{H}, \nu) \to (A, \mathcal{G}, \mu)$ such that $f = \mathrm{U}(p)g$ as needed. ◻

Lemma E.38. For any $\Omega \in \mathbf{EMS}_{\mathrm{std}}$ and $(A, \mathcal{F}, \mu), (B, \mathcal{G}, \nu) \in \mathbf{Prob}_{\mathrm{std}}$ and $\mathbf{EMS}_{\mathrm{std}}$-maps $X : \Omega \to \mathrm{U}(A, \mathcal{F}, \mu)$ and $Y : \Omega \to \mathrm{U}(B, \mathcal{G}, \nu)$, the following are equivalent:

(1) $X = qY$ for some $\mathbf{EMS}_{\mathrm{std}}$-map $q : (B, \mathcal{G}, \nu) \to (A, \mathcal{F}, \mu)$
(2) $X^*(\mathcal{F}, \mu) \mathrel{\overline{\sqsubseteq}} Y^*(\mathcal{G}, \nu)$

Proof. If $X = qY$ then $X^*(\mathcal{F}, \mu) = qY^*(\mathcal{F}, \mu) = Y^* q^*(\mathcal{F}, \mu) \mathrel{\overline{\sqsubseteq}} Y^*(\mathcal{G}, \nu)$ where the final inequality follows from the fact that $q^*(\mathcal{F}, \mu) \mathrel{\overline{\sqsubseteq}} (\mathcal{G}, \nu)$ as probability spaces on $(A, \mathcal{G}, \nu)$. Conversely, if (2) holds then Lemma E.37 gives $q : (B, \mathcal{G}, \nu) \to (A, \mathcal{F}, \mu)$ so $qY = X$. □

Lemma E.39. If $f : (X, \mathcal{F}, \mathcal{N}) \to (Y, \mathcal{G}, \mathcal{M})$ a $\mathbf{EMS}_{\mathrm{std}}$-map and $(\mathcal{H}, \mu), (\mathcal{H}', \mu')$ are two probability spaces on $(Y, \mathcal{G}, \mathcal{M})$, then $(\mathcal{H}, \mu) \mathrel{\overline{\sqsubseteq}} (\mathcal{H}', \mu')$ iff $f^*(\mathcal{H}, \mu) \mathrel{\overline{\sqsubseteq}} f^*(\mathcal{H}', \mu')$.

Proof. The left-to-right implication is straightforward. For the converse, suppose $f^*(\mathcal{H}, \mu) \mathrel{\overline{\sqsubseteq}} f^*(\mathcal{H}', \mu')$. Then $\mathcal{H} \subseteq \mathcal{H}'$, because if $H \in \mathcal{H}$ then $f^{-1}(H) \in f^*\mathcal{H}$, so $f^{-1}(H) = f^{-1}(H') \triangle N$ for some $H' \in \mathcal{H}', N \in \mathcal{N}$ by $f^*\mathcal{H} = f^*\mathcal{H}'$, so $f^{-1}(H) \triangle f^{-1}(H') = f^{-1}(H \triangle H') = N \in \mathcal{N}$, so $H \triangle H' \in \mathcal{N}$ by f negligible-preserving, so $H' = H' \triangle (H \triangle H') \in \mathcal{H}'$ by $\mathcal{H}' \supseteq \mathcal{N}$. And $\mu'|_{\mathcal{H}} = \mu$, because if $H \in \mathcal{H}$ then

$$\begin{aligned}
\mu(H) &= f^*\mu(f^{-1}(H)) \\
&= f^*\mu'(f^{-1}(H)) \text{ by } f^*\mu = f^*\mu'|_{f^*\mathcal{H}} \\
&= f^*\mu(f^{-1}(H') \triangle N) \text{ for } H' \in \mathcal{H}', N \in \mathcal{N} \text{ as above} \\
&= \mu'(H') \\
&= \mu'(H' \triangle (H \triangle H')) \text{ where } H \triangle H' \in \mathcal{N} \text{ as above} \\
&= \mu'(H') \text{ because } \mathcal{N} = \mathrm{negligibles}(\mu').
\end{aligned}$$

□

Lemma E.40. The ordering $\overline{\sqsubseteq}$ is a map $\overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \mathrm{Prop}$ of absolutely continuous sets corresponding to the map $\sqsubseteq : \mathbb{P} \times \mathbb{P} \to \mathrm{Prop}$ of enhanced measurable sheaves across the equivalence in Theorem E.13.

Proof. For any $\Omega \in \mathbf{EMS}_{\mathrm{std}}$ and $\mathbf{EMS}_{\mathrm{std}}$-map $p : \mathbb{I}^\omega \to \Omega$ with finite footprint, two elements $[(A, \mathcal{F}, \mu), X]$ and $[(B, \mathcal{G}, \nu), Y]$ of $\mathbb{P}\Omega$ are related by $\sqsubseteq_\Omega$ iff $X = qY$ for some $\mathbf{EMS}_{\mathrm{std}}$-map $q : (B, \mathcal{G}, \nu) \to (A, \mathcal{F}, \mu)$, iff $X^*(\mathcal{F}, \mu) \mathrel{\overline{\sqsubseteq}} Y^*(\mathcal{G}, \nu)$ by Lemma E.38, iff $(Xp)^*(\mathcal{F}, \mu) \mathrel{\overline{\sqsubseteq}} (Yp)^*(\mathcal{G}, \nu)$ by Lemma E.39, and $(Xp)^*(\mathcal{F}, \mu)$ and $(Yp)^*(\mathcal{G}, \nu)$ are the standardizable probability spaces on $\mathbb{I}^\omega$ corresponding to $[A, X]$ and $[B, Y]$ across the equivalence in Theorem E.13. □

Definition E.41. Let $\mathbb{P}^2_\perp$ be the $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}} \mathbb{I}^\omega$-set of pairs $((\mathcal{G}, \mu), (\mathcal{H}, \nu)) \in \overline{\mathbb{P}} \times \overline{\mathbb{P}}$ for which $(\mathcal{G}, \mu)$ and $(\mathcal{H}, \nu)$ are independently combinable, with action inherited from $\overline{\mathbb{P}} \times \overline{\mathbb{P}}$.

Lemma E.42. If $(\mathcal{H}, \mu)$ and $(\mathcal{K}, \nu)$ are independently combinable on a standard enhanced measurable space $(Y, \mathcal{G}, \mathcal{M})$ and f is a $\mathbf{EMS}_{\mathrm{std}}$-map $(X, \mathcal{F}, \mathcal{N}) \to (Y, \mathcal{G}, \mathcal{M})$, then $f^*(\mathcal{H}, \mu)$ and $f^*(\mathcal{K}, \nu)$ are independently combinable on $(X, \mathcal{F}, \mathcal{N})$.

Proof. Suppose $(\mathcal{L}, \rho)$ witnesses the independent combinability of $(\mathcal{H}, \mu)$ and $(\mathcal{K}, \nu)$, so $\mathcal{H} \subseteq \mathcal{L} \supseteq \mathcal{K}$ and $\mu = \rho|_{\mathcal{H}}$ and $\nu = \rho|_{\mathcal{K}}$ and $\rho(H \cap K) = \rho(H)\rho(K)$ for all $H \in \mathcal{H}, K \in \mathcal{K}$. It is straightforward to show $f^*\mathcal{H} \subseteq f^*\mathcal{L} \supseteq f^*\mathcal{K}$ and $f^*\mathcal{H} = f^*\mathcal{L}|_{f^*\mathcal{H}}$ and $f^*\mathcal{K} = f^*\mathcal{L}|_{f^*\mathcal{K}}$. Fix arbitrary $f^{-1}(H) \triangle N \in f^*\mathcal{H}$ and $f^{-1}(K) \triangle N' \in f^*\mathcal{K}$ for $H \in \mathcal{H}$ and $K \in \mathcal{K}$ and $N, N' \in \mathcal{N}$. Since $\cap$ distributes over $\triangle$,

$$\begin{aligned}
f^*\rho((f^{-1}(H) \triangle N) \cap (f^{-1}(K) \triangle N')) &= f^*\rho((f^{-1}(H) \cap f^{-1}(K)) \triangle \underbrace{(f^{-1}(H) \cap N') \triangle (N \cap f^{-1}(K)) \triangle (N \triangle N')}_{\in \mathcal{N}}) \\
&= f^*\rho(f^{-1}(H \cap K)) \text{ because } \mathcal{N} = \mathrm{negligibles}(f^*\rho) \\
&= \rho(H \cap K) \\
&= \rho(H)\rho(K) \\
&= f^*\rho(f^{-1}(H) \triangle N) f^*\rho(f^{-1}(K) \triangle N')
\end{aligned}$$

so $f^*\mathcal{H}$ and $f^*\mathcal{K}$ are independent sub-σ-algebras in $(X, f^*\mathcal{L}, \rho)$. □

Lemma E.43. For any $\Omega \in \mathbf{EMS}_{\mathrm{std}}$ and $(A, \mathcal{F}, \mu), (B, \mathcal{G}, \nu) \in \mathbf{Prob}_{\mathrm{std}}$ and $\mathbf{EMS}_{\mathrm{std}}$-maps $X : \Omega \to \mathrm{U}(A, \mathcal{F}, \mu)$ and $Y : \Omega \to \mathrm{U}(B, \mathcal{G}, \nu)$, the following are equivalent:

(1) The pairs $((A, \mathcal{F}, \mu), X)$ and $((B, \mathcal{G}, \nu), Y)$ factor through a tensor product; i.e., there exist Ω_1, Ω_2 and $f : \Omega \to \Omega_1 \otimes \Omega_2$ and $X' : \Omega_1 \to \mathrm{U}(A, \mathcal{F}, \mu)$ and $Y' : \Omega_2 \to \mathrm{U}(B, \mathcal{G}, \nu)$ with $X = X' \ \mathrm{fst}\ f$ and $Y = Y' \ \mathrm{snd}\ f$.
(2) The pullbacks $X^*(\mathcal{F}, \mu)$ and $Y^*(\mathcal{G}, \nu)$ are independently combinable.

Moreover, in the case where both hold, the pullback $((X' \otimes Y')f)^*(\mathcal{F} \otimes \mathcal{G}, \mu \otimes \nu)$ of the tensor product $(A, \mathcal{F}, \mu) \otimes (B, \mathcal{G}, \nu)$ along the composite $\left(\Omega \xrightarrow{f} \Omega_1 \otimes \Omega_2 \xrightarrow{X' \otimes Y'} A \otimes B\right)$ is the independent combination of $X^*(\mathcal{F}, \mu)$ and $Y^*(\mathcal{G}, \nu)$.

Proof. Suppose (1). Form the tensor product $(A\times B, \mathcal{F}\otimes\mathcal{G}, \mu\otimes\nu)$ of $(A,\mathcal{F},\mu)$ and $(B,\mathcal{G},\nu)$, so that $X = \mathrm{fst}(X'\otimes Y')f$ and $Y = \mathrm{snd}(X'\otimes Y')f$. By construction the sub-σ-algebras $\mathrm{fst}^*\mathcal{F}$ and $\mathrm{snd}^*\mathcal{G}$ are independent in the tensor product and the measure $\mu\otimes\nu$ restricts to $\mathrm{fst}^*\mu$ on $\mathrm{fst}^*\mathcal{F}$ and $\mathrm{snd}^*\nu$ on $\mathrm{snd}^*\mathcal{G}$, so $\mathrm{fst}^*(\mathcal{F},\mu)$ and $\mathrm{snd}^*(\mathcal{G},\nu)$ are independently combinable in $(A\times B, \mathcal{F}\otimes\mathcal{G}, \mathrm{negligibles}(\mu\otimes\nu))$. Hence $X^*(\mathcal{F},\mu) = (\mathrm{fst}(X'\otimes Y')f)^*(\mathcal{F},\mu)$ and $Y^*(\mathcal{G},\nu) = (\mathrm{snd}(X'\otimes Y')f)^*(\mathcal{G},\nu)$ are independently combinable in Ω by Lemmas E.25 and E.42.

Conversely, suppose (2), and let $(\mathcal{H},\rho)$ be the standardizable probability space on Ω witnessing independent combinability of $X^*(\mathcal{F},\mu)$ and $Y^*(\mathcal{G},\nu)$. There is a $\mathbf{Prob}_{\mathrm{std}}$-map $g : (\Omega,\mathcal{H},\rho) \to (A,\mathcal{F},\mu)\otimes(B,\mathcal{G},\nu)$ defined by $f(\omega) = (X(\omega), Y(\omega))$, measure-preserving because $\rho(g^{-1}(F\times G)) = \rho(X^{-1}(F)\cap Y^{-1}(G)) = \rho(X^{-1}(F))\rho(Y^{-1}(G)) = \mu(F)\nu(G) = (\mu\otimes\nu)(F\times G)$ for all rectangles $F\times G \in \mathcal{F}\otimes\mathcal{G}$. Moreover, g satisfies $\mathrm{fst}\ g = X$ and $\mathrm{snd}\ g = Y$, so $\mathrm{U}g$ is a $\mathbf{EMS}_{\mathrm{std}}$-map showing that $[A,X]$ and $[B,Y]$ factor through the tensor product $(A,\mathcal{F},\mathrm{negligibles}(\mu))\otimes(B,\mathcal{G},\mathrm{negligibles}(\nu))$.

It only remains to show, in case (1) and (2) both hold, that $((X'\otimes Y')f)^*(\mathcal{F}\otimes\mathcal{G},\mu\otimes\nu)$ is the independent combination; i.e., that it is the $\overline{\sqsubseteq}$-smallest standardizable probability space on Ω witnessing the independent combinability of $X^*(\mathcal{F},\mu)$ and $Y^*(\mathcal{G},\nu)$. Suppose $(\mathcal{H},\rho)$ is another such witness. The map $g : (\Omega,\mathcal{H},\rho) \to (A,\mathcal{F},\mu)\otimes(B,\mathcal{G},\nu)$ constructed above is equal to $(X'\otimes Y')f$ as a set-function, since $\mathrm{fst}\ g = X = \mathrm{fst}(X'\otimes Y')f$ and $\mathrm{snd}\ g = Y = \mathrm{snd}(X'\otimes Y')f$. This shows $(X'\otimes Y')f$ is measure-preserving as a map with domain $(\Omega,\mathcal{H},\rho)$, so $((X'\otimes Y')f)^*(\mathcal{F}\otimes\mathcal{G},\mu\otimes\nu)\ \overline{\sqsubseteq}\ (\mathcal{H},\rho)$ as required. □

Lemma E.44. The $\mathrm{Aut}_{\mathbf{EMS}_{\mathrm{std}}}\mathbb{I}^\omega$-set $\mathbb{P}^2_\perp$ is an absolutely continuous set corresponding to the enhanced measurable sheaf $\mathbb{P}\otimes\mathbb{P}$ across the equivalence in Theorem E.13.

Proof. By Lemma D.9, each $(\mathbb{P}\otimes\mathbb{P})(\Omega)$ is a subset of $\mathbb{P}\Omega\times\mathbb{P}\Omega$ consisting of the pairs of probability spaces on Ω that factor through a tensor product. Thus the corresponding absolutely continuous set is a subset of the absolutely continuous set $\overline{\mathbb{P}}\times\overline{\mathbb{P}}$ of pairs of standardizable probability spaces with finite footprint. To determine this subset, it suffices to compute the image of the inclusion $\mathbb{P}\otimes\mathbb{P}\hookrightarrow\mathbb{P}\times\mathbb{P}$ across the equivalence in Theorem E.13. For any $\Omega\in\mathbf{EMS}_{\mathrm{std}}$ and $\mathbf{EMS}_{\mathrm{std}}$-map $p : \mathbb{I}^\omega \to \Omega$ with finite footprint, this inclusion sends a pair $((\mathcal{G},\mu),(\mathcal{H},\nu))$ of probability spaces on Ω that factor through a tensor product to $(p^*(\mathcal{G},\mu), p^*(\mathcal{H},\nu))$, a pair of independently-combinable probability spaces on $\mathbb{I}^\omega$ with finite footprint. This hits every pair of independently-combinable probability spaces with finite footprint by Lemma E.43, so the image of the inclusion $\mathbb{P}\otimes\mathbb{P}\hookrightarrow\mathbb{P}\times\mathbb{P}$ corresponds to $\mathbb{P}^2_\perp$ as claimed. □

Definition E.45. Let $(\mathcal{G},\mu)$ and $(\mathcal{H},\nu)$ be two standardizable probability spaces on $\mathbb{I}^\omega$ with finite footprint. If $(\mathcal{G},\mu)$ and $(\mathcal{H},\nu)$ are independently combinable, let $\overline{\mathrm{join}}((\mathcal{G},\mu),(\mathcal{H},\nu))$ be their independent combination.

Lemma E.46. Let $f : (X,\mathcal{F},\mathcal{N}) \to (Y,\mathcal{G},\mathcal{M})$ be a $\mathbf{EMS}_{\mathrm{std}}$-map. If $(\mathcal{L},\rho)$ is the independent combination of $(\mathcal{H},\mu)$ and $(\mathcal{K},\nu)$ then $f^*(\mathcal{L},\rho)$ is the independent combination of $f^*(\mathcal{H},\mu)$ and $f^*(\mathcal{K},\nu)$.

Proof. Let $h : Y \to \mathrm{U}A$ and $k : Y \to \mathrm{U}B$ be the $\mathbf{EMS}_{\mathrm{std}}$-maps witnessing standardizability of $(\mathcal{H},\mu)$ and $(\mathcal{K},\nu)$, so $(\mathcal{H},\mu) = h^*A$ and $(\mathcal{K},\nu) = k^*B$. By Lemma E.43, the independent combination $(\mathcal{L},\rho)$ is the pullback $p^*(A\otimes B)$, where $p : Y \to \mathrm{U}(A\otimes B)$ is the map defined by $p(y) = (h(y),k(y))$. Now $f^*(\mathcal{H},\mu)$ and $f^*(\mathcal{K},\nu)$ arise via pullback from hf and kf respectively, so by Lemma E.43 again their independent combination is $q^*(A\otimes B)$ where $q : X \to \mathrm{U}(A\otimes B)$ is the map defined by $q(x) = (h(f(x)),k(f(x)))$. By definition $q = pf$, so $q^*(A\otimes B) = (pf)^*(A\otimes B) = f^*p^*(A\otimes B) = f^*(\mathcal{L},\rho)$ as claimed. □

Theorem E.47. The operation $\overline{\mathrm{join}}$ is a map $\mathbb{P}^2_\perp \to \overline{\mathbb{P}}$ of absolutely continuous sets corresponding to the map $\mathrm{join} : \mathbb{P}\otimes\mathbb{P} \to \mathbb{P}$ of enhanced measurable sheaves across the equivalence in Theorem E.13.

Proof. Fix $\Omega\in\mathbf{EMS}_{\mathrm{std}}$ and $p : \mathbb{I}^\omega \to \Omega$ a $\mathbf{EMS}_{\mathrm{std}}$-map with finite footprint. Let $((\mathcal{F}',\mu'),(\mathcal{G}',\nu'))\in\mathbb{P}\otimes\mathbb{P}$ be a pair of standardizable probability spaces on Ω that factor through a tensor product, so $(\mathcal{F}',\mu') = (X\ \mathrm{fst}\ f)^*(\mathcal{F},\mu)$ and $(\mathcal{G}',\nu') = (Y\ \mathrm{snd}\ f)^*(\mathcal{G},\nu)$ for standard probability spaces $(A,\mathcal{F},\mu)$, $(B,\mathcal{G},\nu)$ and standard enhanced measurable spaces Ω_1,Ω_2 and $\mathbf{EMS}_{\mathrm{std}}$-maps $f : \Omega\to\Omega_1\otimes\Omega_2$ and $X : \Omega_1 \to A$ and $Y : \Omega_1 \to B$. The map join sends $((\mathcal{F}',\mu'),(\mathcal{G}',\nu'))$ to $((X\otimes Y)f)^*(\mathcal{F}\otimes\mathcal{G},\mu\otimes\nu)$ by Note D.11, their independent combination by Lemma E.43, so $p^*((X\otimes Y)f)^*(\mathcal{F}\otimes\mathcal{G},\mu\otimes\nu)$ is the independent combination of $p^*(\mathcal{F}',\mu')$ and $p^*(\mathcal{G}',\nu')$ by Lemma E.46. Putting this together gives $p^*\mathrm{join}((\mathcal{F}',\mu'),(\mathcal{G}',\nu')) = \overline{\mathrm{join}}(p^*(\mathcal{F}',\mu'),p^*(\mathcal{G}',\nu'))$, so join corresponds to $\overline{\mathrm{join}}$ across the equivalence in Theorem E.13 as claimed. □

Theorem E.48. The PDM $(\mathbb{P},\mathrm{join},\mathrm{emp},\sqsubseteq)$ internal to enhanced measurable sheaves corresponds to the PDM $(\overline{\mathbb{P}},\overline{\mathrm{join}},\overline{\mathrm{emp}},\overline{\sqsubseteq})$ internal to absolutely continuous sets across the equivalence in Theorem E.13.

Proof. Theorem E.31 shows $\mathbb{P}$ corresponds to $\overline{\mathbb{P}}$, Theorem E.47 shows join corresponds to $\overline{\mathrm{join}}$, Lemma E.36 shows emp corresponds to $\overline{\mathrm{emp}}$, and Lemma E.40 shows $\sqsubseteq$ corresponds to $\overline{\sqsubseteq}$. □