A Nominal Approach to Probabilistic Separation Logic

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$$X \leftarrow \texttt{flip}\ 1/2;$$

$$Y \leftarrow \texttt{flip}\ 1/2;$$

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↑

X and Y are independent random variables

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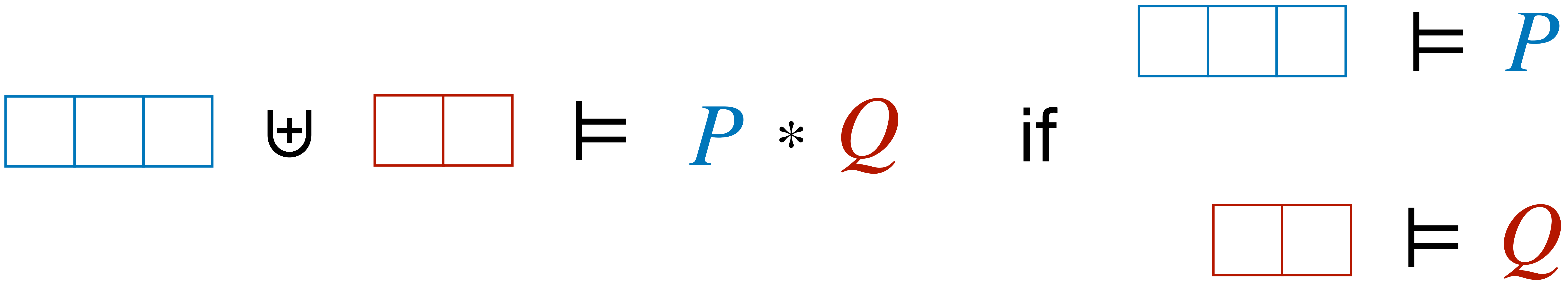

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- Separate probability spaces into independent subspaces:

$$(\mathcal{F}, \mu) \bullet (\mathcal{G}, \nu) \vDash P * Q \quad \text{if} \quad \begin{matrix} (\mathcal{F}, \mu) \vDash P \\ (\mathcal{G}, \nu) \vDash Q \end{matrix}$$

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- Separate probability spaces into independent subspaces:

$$(\mathscr{F}, \mu) \bullet (\mathscr{G}, \nu) \models P * Q \quad \text{if} \quad \begin{array}{l} (\mathscr{F}, \mu) \models P \\ (\mathscr{G}, \nu) \models Q \end{array}$$

$\mathscr{F}, \mathscr{G}$ are σ-algebras,
μ, ν are probability measures

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independent combination
("disjoint union for spaces")

?!

?!

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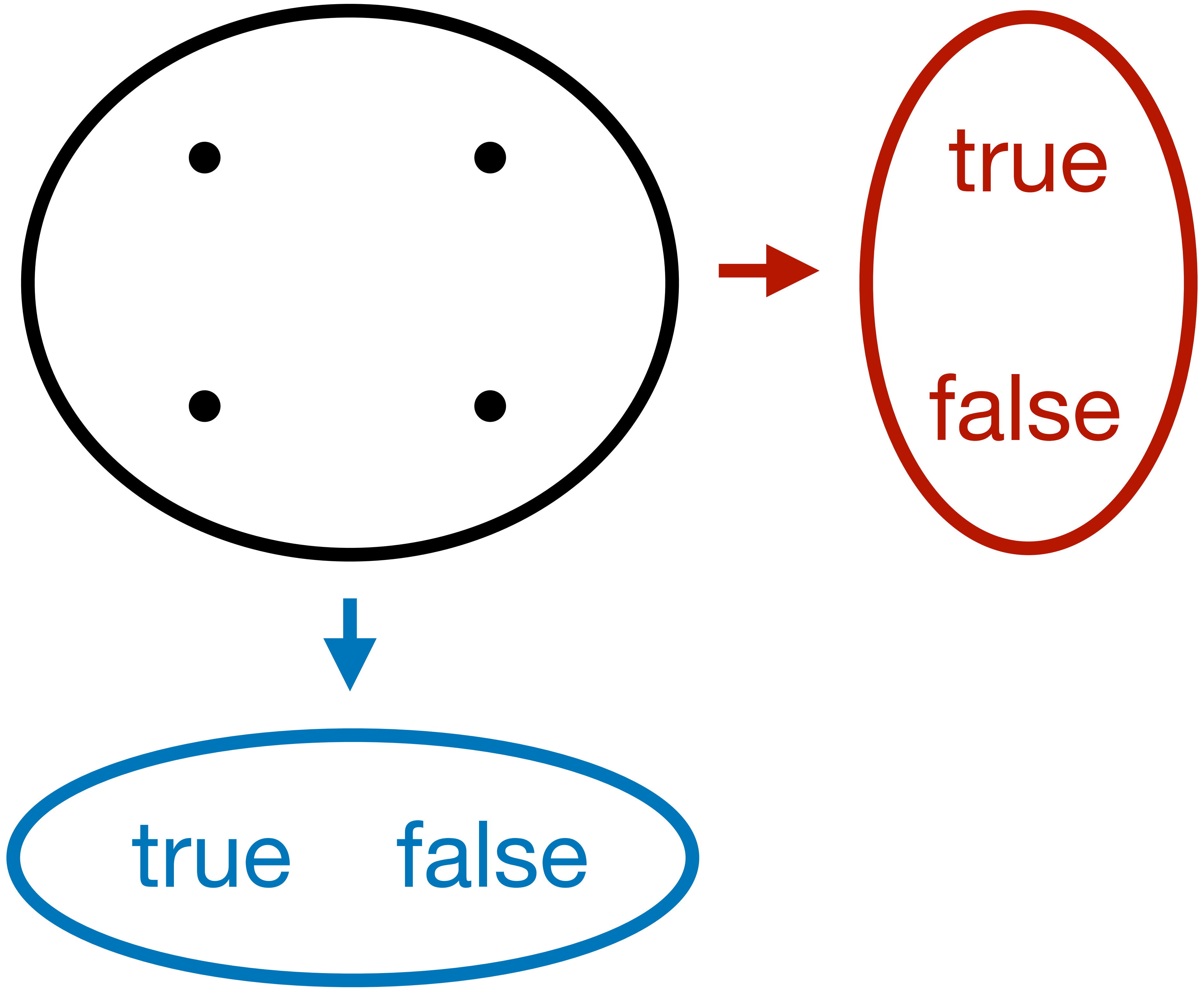

This paper: a nominal answer

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"enhanced measurable sheaves"

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$\mathbf{EMS}$

"enhanced measurable sheaves"

$\mathbf{Set}^{\ll}$

"absolutely continuous sets"

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independence via product spaces	$\sim$	Lilac's independent combination, for discrete probability

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separation = heaps
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A more abstract view: resources to atomic sheaves

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$\mathbf{R}$

symmetric monoidal

"resources"

A more abstract view: resources to atomic sheaves

$\mathbf{R}$

$\mathbf{S}$

symmetric monoidal, atomic, ...

"resource shapes"

A more abstract view: resources to atomic sheaves

$$|\cdot| \quad : \quad \mathbf{R} \xrightarrow{\text{monoidal}} \mathbf{S}$$

"forget the contents of the resource"

A more abstract view: resources to atomic sheaves

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In $\mathrm{Sh}_{\mathrm{atomic}}(\mathbf{S})$,

$\mathrm{Res} = \varinjlim_{r:\mathrm{Core}(\mathbf{R})} \mathbf{S}(-, |r|)$ is a sheaf of resources

$\mathrm{Res} \otimes \mathrm{Res}$ is a sheaf of separated resources

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Lemma C.23

A more abstract view: atomic sheaves to G-sets

- Under suitable conditions, one can find an object s_∞ that produces an equivalence $i : \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{S}) \simeq \mathrm{Aut}(s_\infty)$-sets.

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- This equivalence gives a correspondence

$$\begin{array}{ccc} \mathrm{Res} \otimes \mathrm{Res} & & i(\mathrm{Res} \otimes \mathrm{Res}) \\ \text{in} & \sim & \text{in} \\ \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{S}) & & \mathrm{Aut}(s_\infty)\text{-sets.} \end{array}$$

Our probabilistic analog: the discrete case

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discrete independent combination comes from the monoidal structure on $\mathbf{Prob}_{\leq\omega}$

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Theorem 3.21

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Lilac's independent combination comes from the monoidal structure on $\mathbf{Prob}_{\mathrm{std}}$

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Theorem 4.24

See the paper for...

- Precise definitions
- Separation logic details
- Constructing suitable s_∞s
- Properties of $\mathbf{EMS}_{\mathrm{std}}$ (monoidal, atomic, subcanonical)

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Probabilistic Programming Semantics for Name Generation

MARCIN SABOK, McGill University, Canada
SAM STATON, University of Oxford, United Kingdom
DARIO STEIN, University of Oxford, United Kingdom
MICHAEL WOLMAN, McGill University, Canada

Equivalence and Conditional Independence in Atomic Sheaf Logic

Alex Simpson*

Probability Sheaves and the Giry Monad*

Alex Simpson

li.john@northeastern.edu

Thanks!

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"heap shapes"

The folklore: separation logic in **Sch**

- $\mathbf{Sch} = \mathrm{Sh}_{\mathrm{atomic}}(\mathbf{FinInj}^{\mathrm{op}})$
- $(\mathbf{FinInj}^{\mathrm{op}}, +, \varnothing)$ is a monoidal category.
- Yields a monoidal structure $(\otimes, \mathrm{I})$ on $\mathbf{Sch}$, by Day convolution.

The folklore: separation logic in **Sch**

- There is a sheaf of heaps $\mathbb{H}(L) = L \rightharpoonup_{\mathrm{fin}} \mathbb{Z}$.
- The convolution $\mathbb{H} \otimes \mathbb{H}$ is a sheaf of separated heaps:

$$(\mathbb{H} \otimes \mathbb{H})(L) = \{(h, h') \mid \mathrm{dom}(h) \cap \mathrm{dom}(h') = \varnothing\}$$

- These form the basic ingredients of separation logic.

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The folklore: separation logic in **Nom**

- $\mathbf{Nom} = \mathbf{G}\ \mathcal{S}\mathrm{et}$, where $\mathbf{G} = \mathrm{Aut}_{\mathrm{fin}}(\mathbb{N})$ + a particular topology.
- Heaps: $\overline{\mathbb{H}} = \mathbb{N} \rightharpoonup_{\mathrm{fin}} \mathbb{Z}$
- Separated heaps:

$$\overline{\mathbb{H}}_{\mathrm{sep}} = \{(h, h') \in \overline{\mathbb{H}} \times \overline{\mathbb{H}} \mid \mathrm{dom}(h) \cap \mathrm{dom}(h') = \varnothing\}$$

- These again form the basic ingredients of separation logic.

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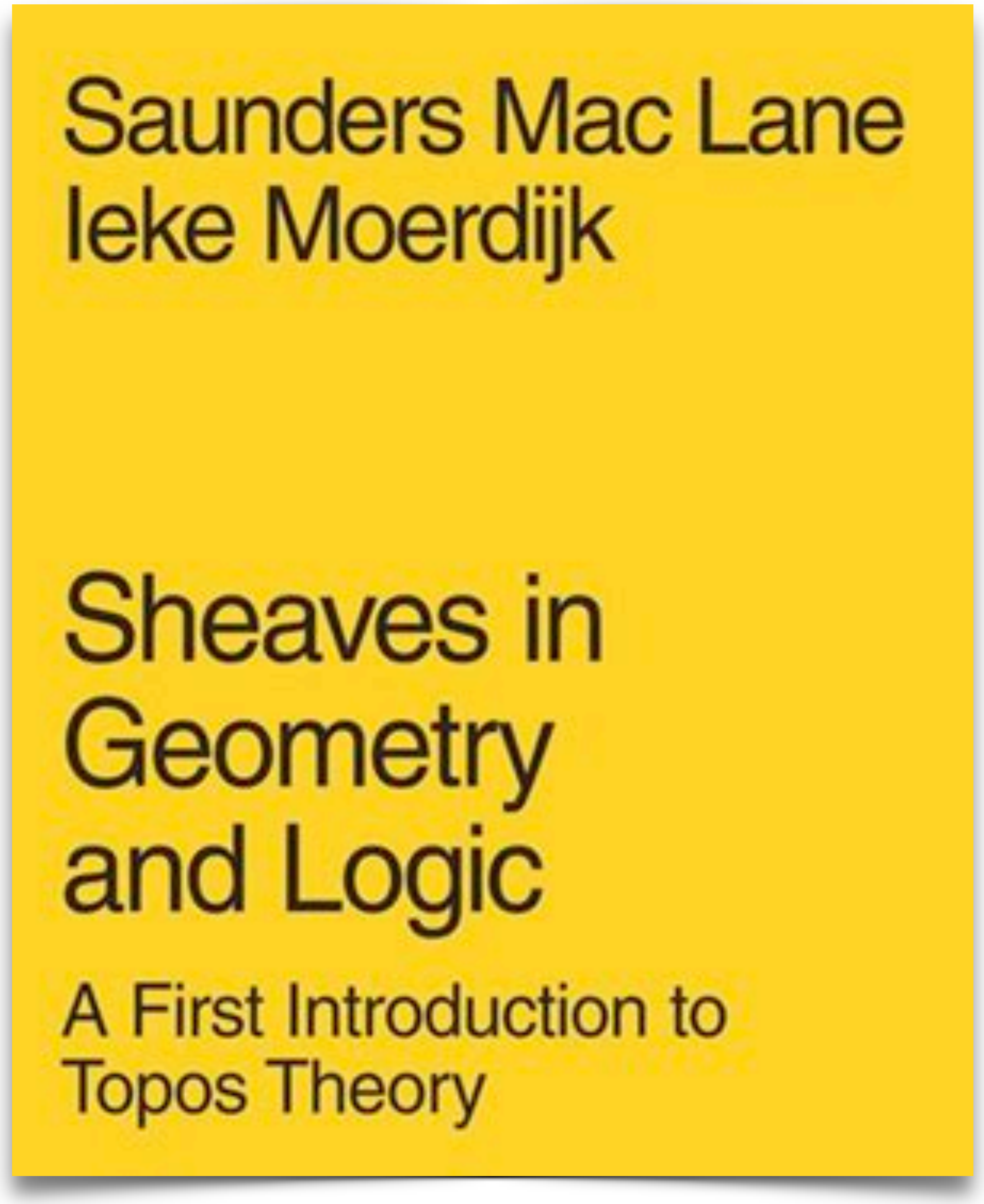

, Theorem III.9.2: $\mathbf{Sch} \simeq \mathbf{Nom}$.

The folklore: the equivalence

- Across this equivalence,

$\mathbf{Sch}$		$\mathbf{Nom}$
$\mathbb{H}$	corresponds to	$\overline{\mathbb{H}}$
$\mathbb{H} \otimes \mathbb{H}$	corresponds to	$\overline{\mathbb{H}}_{\mathrm{sep}}$

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- Key idea: every renaming can be implemented by a permutation

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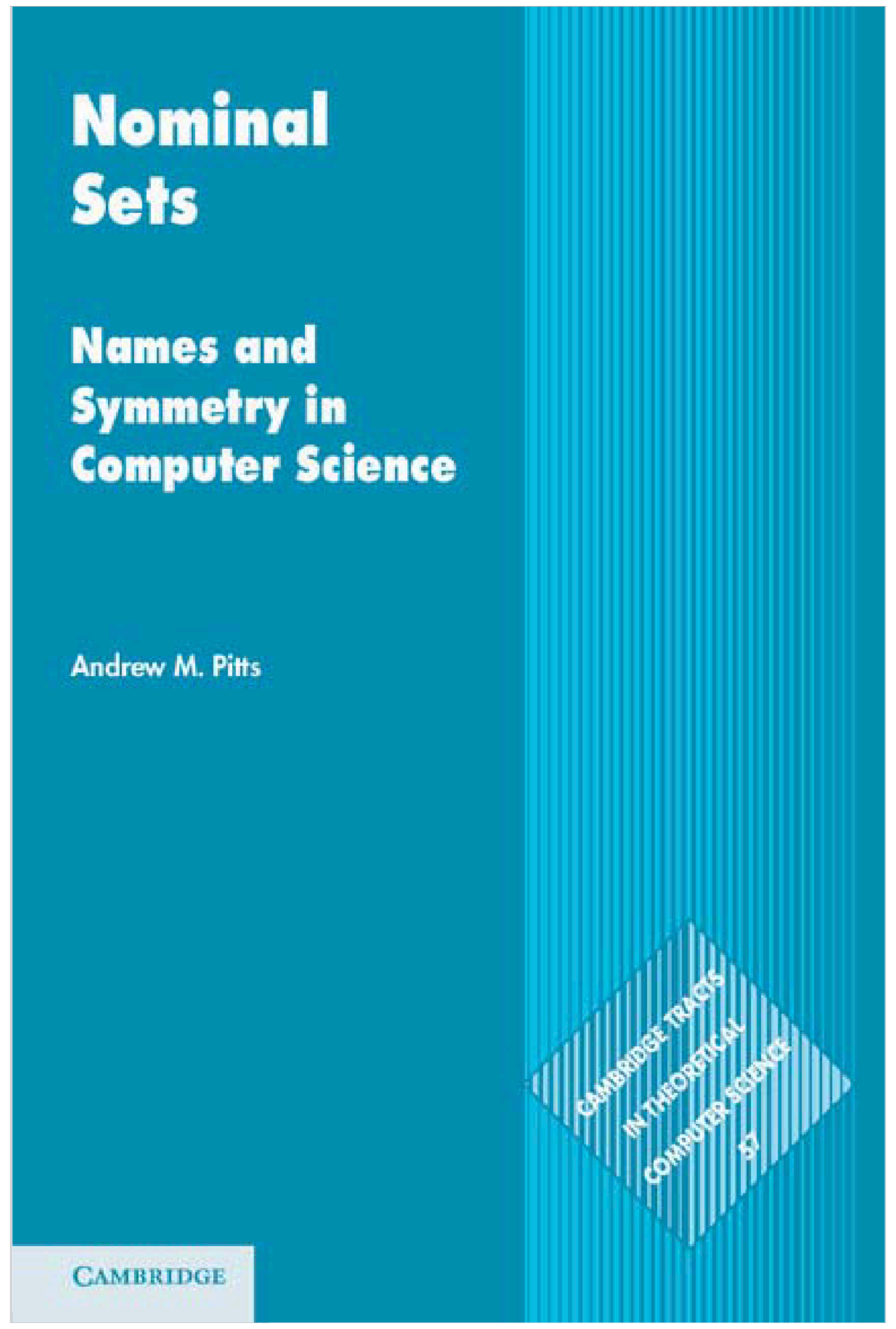
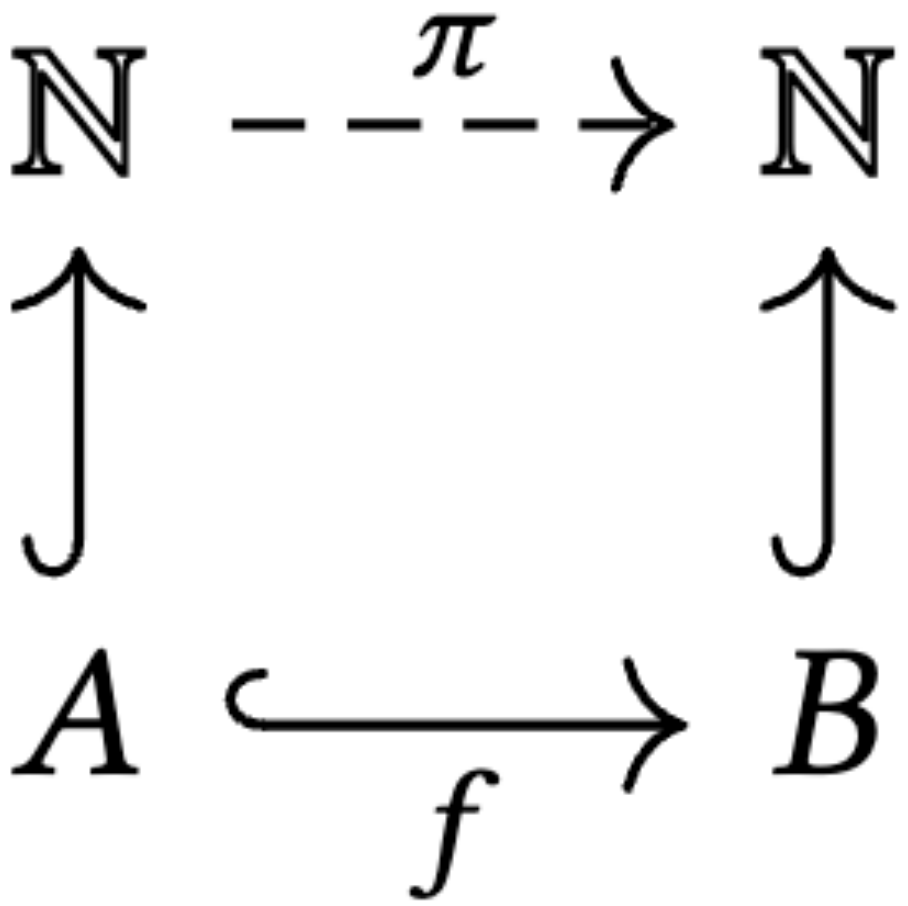


, Lemma 1.14 (Homogeneity):



Our probabilistic analog: the discrete case

- Key lemma:

$$
\begin{array}{ccc}
\mathbb{N} & \overset{\pi}{\dashrightarrow} & \mathbb{N} \\
\uparrow & & \uparrow \\
A & \underset{f}{\hookrightarrow} & B
\end{array}
\quad \text{becomes} \quad
\begin{array}{ccc}
[0,1] & \overset{\pi}{\dashrightarrow} & [0,1] \\
\downarrow & & \downarrow \\
X & \underset{f}{\twoheadrightarrow} & Y
\end{array}
$$

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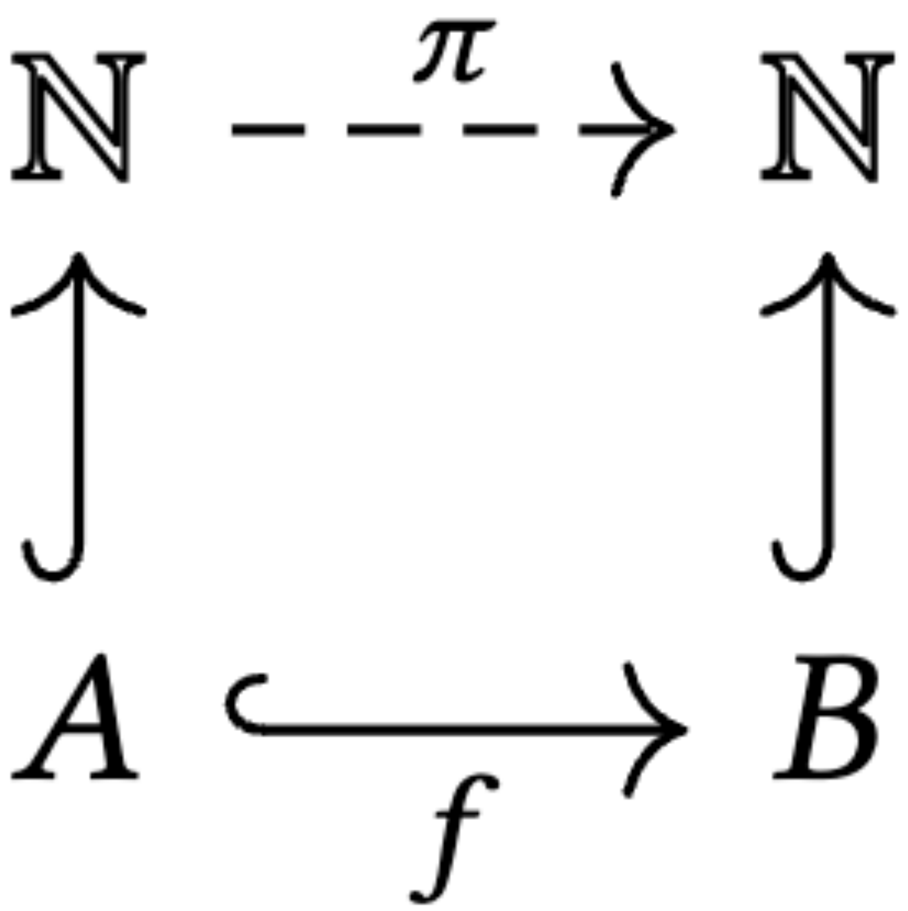

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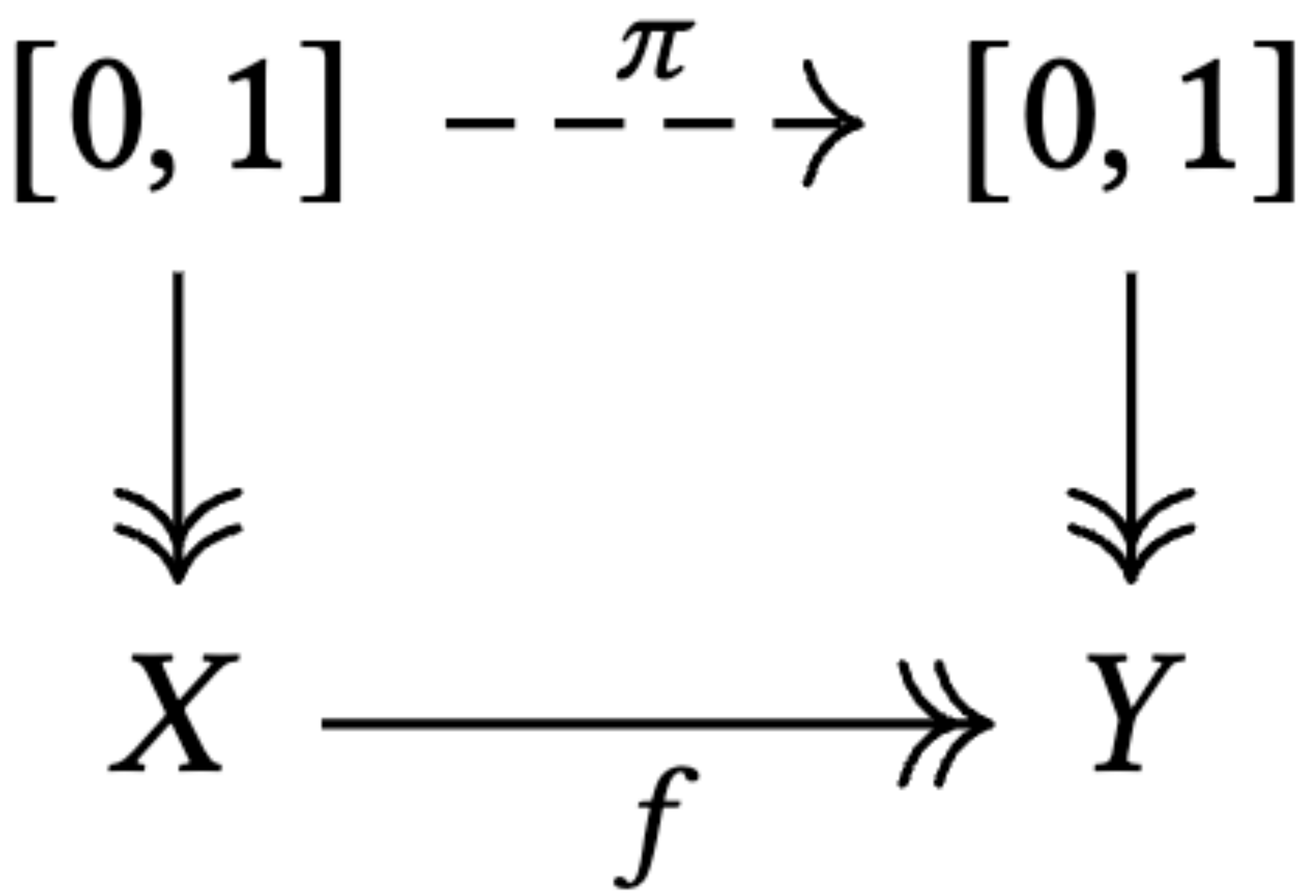

- Proof roughly boils down to: any two nonnegligible measurable subsets of $[0,1]$ are measurably isomorphic.

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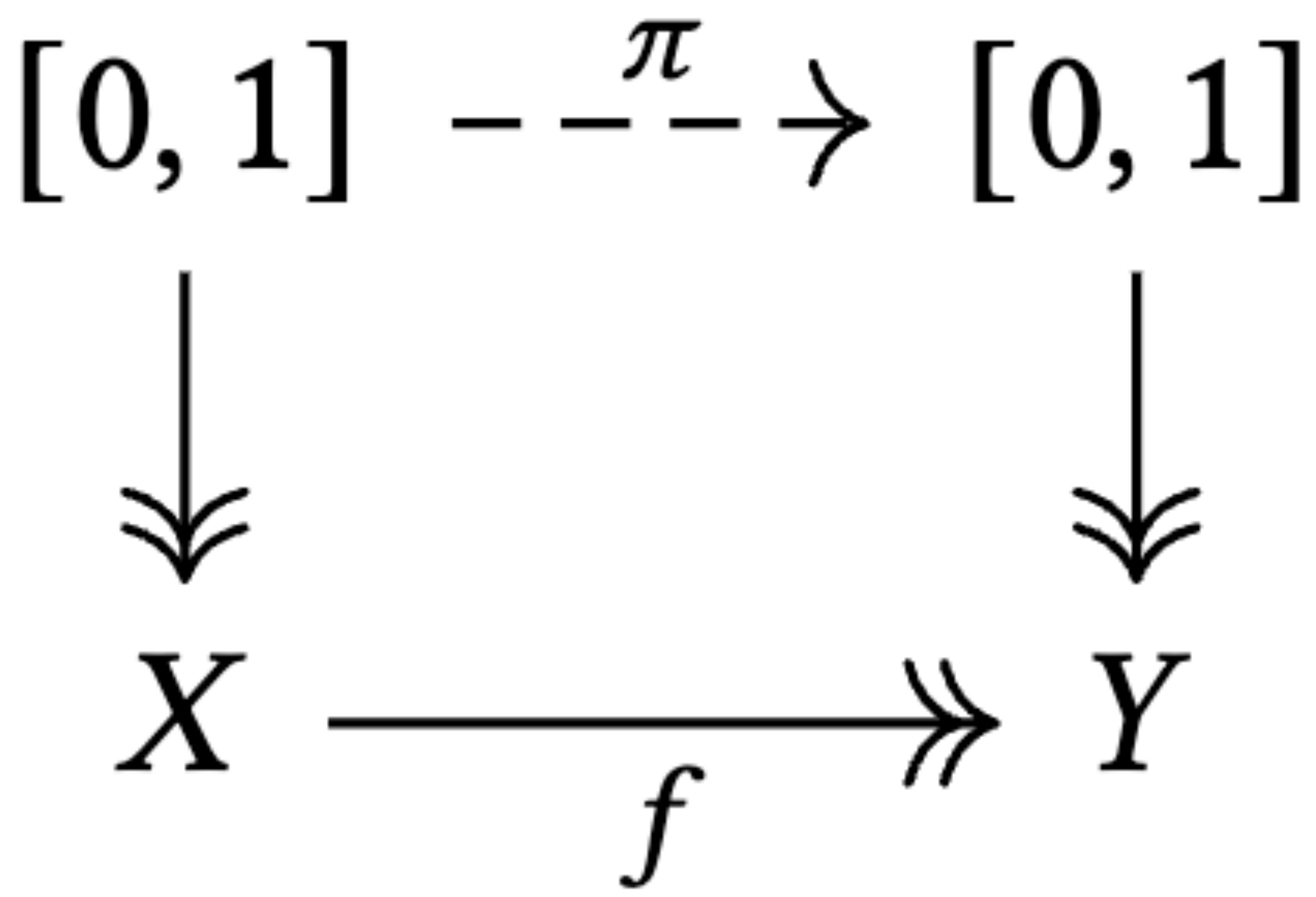

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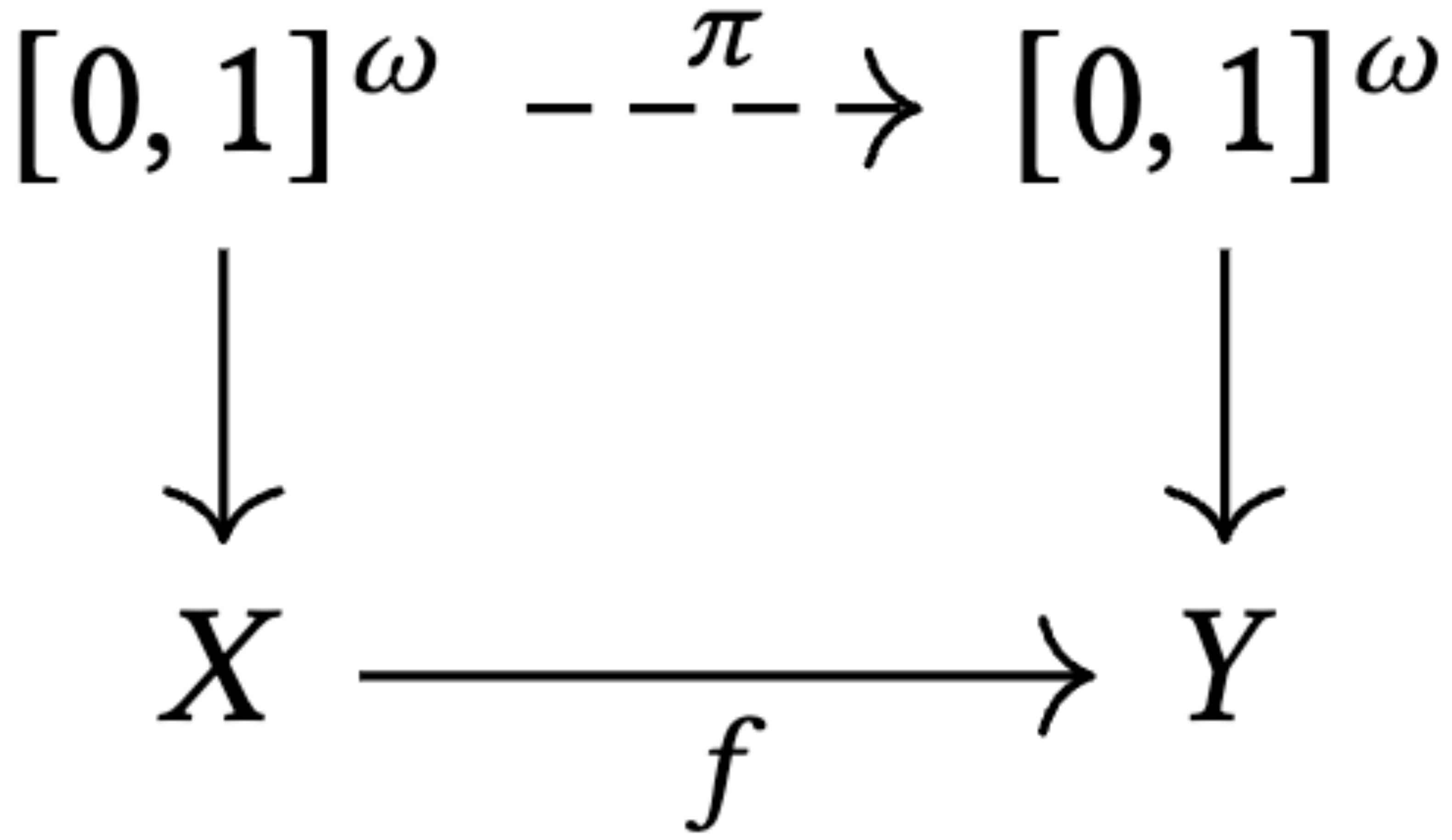

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- Key lemma:

$$\begin{array}{ccc} [0,1] & \overset{\pi}{\dashrightarrow} & [0,1] \\ \big\downarrow & & \big\downarrow \\ X & \underset{f}{\twoheadrightarrow} & Y \end{array} \qquad \text{becomes} \qquad \begin{array}{ccc} [0,1]^{\omega} & \overset{\pi}{\dashrightarrow} & [0,1]^{\omega} \\ \big\downarrow & & \big\downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

- Proof requires some heavy-duty measure theory.